2. SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

2.1 Ordinary 2nd Order Linear Differential Equations

2.1.1 Origin of Differential Equations: the Harmonic Oscillator as an Example

We consider a particle of mass m that is moving along a straight line in x–direction. At time t, its coordinate is $x = x(t)$. It is attached to springs with spring constant $k > 0$ so that there is a 'restoring' force $f_r(x) = -kx$ acting on the particle. At $x = 0$, the mass is in equilibrium and no force is acting. In addition, there is a friction force $f_f(v) = -\gamma v$ acting on the particle which is proportional (with friction constant $\gamma > 0$) to its velocity $v = \dot{x}(t)$, and an external force $f_e(x)$ that could have its origin in, e.g., some crazy experimentalist fiercly forcing the mass to follow her hand.

Newton's law states that $m\ddot{x}(t)$ equals the sum $f_r(x)+f_f(x)+f_e(x)$ of all forces on the particle, i.e.

$$
\begin{aligned}
m\ddot{x}(t) &= -kx - \gamma\dot{x}(t) + f_e(x) \Leftrightarrow \\
\ddot{x}(t) + \frac{\gamma}{m}\dot{x}(t) + \frac{k}{m}x(t) &= \frac{1}{m}f_e(x), \quad k > 0, \gamma > 0. \qquad (2.1)
\end{aligned}
$$

To find the position x of the particle at time t, i.e. the function $x(t)$, we have to solve the **differential equation of the forced, damped linear harmonic oscillator**, Eq. (2.1). Learn this standard form of the forced damped harmonic oscillator by heart and it will save you from much misery in the future.
CHECK: to which forces do the terms 'forced', 'damped', and 'harmonic' refer ?

Is this a well–defined task? No, in order to know $x(t)$ at all times later than, say, $t = 0$, we must specify the **initial conditions**, i.e. the initial position of the particle $x(t = 0)$ and its initial velocity $\dot{x}(t = 0)$.

Eq. (2.1)is called **2nd order differential equation** because the highest derivative appearing is a second derivative. Because Newton's law (for a general force) leads to second derivatives (acceleration term!), 2nd order differential equations belong to the most important differential equations in physics.

Eq. (2.1) is called **linear** because we don't have terms like $\ddot{x}^2(t)$ or $x^4(t)$. In general and in more complicated cases (e.g., motion in three dimensions), such terms can leads to **chaos**. The study of differential equations therefore is of paramount importance in order to understand chaos.

Eq. (2.1) is called **Ordinary** because the desired function x is a function of one variable (t) only and not more than one variable, in which case differential equations are called partial differential equations.

2.1.2 Definitions

In the mathematic literature, people sometimes don't care about the physical background of equations and introduce other notations. In the following, instead of $x(t)$, $\dot{x}(t)$ etc we discuss differential equations for functions $y(x)$ of one variable x, with $y'(x)$ denoting the first and $y''(x)$ the second derivative, respectively.
A 2nd order inhomogeneous linear differential equation for the function $y(x)$ has the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x), \tag{2.2}$$

where $p(x)$, $q(x)$, and $f(x)$ are known functions of x and $y(x)$ is the function one would like to calculate.

In general, there is no method to obtain a solution $y(x)$ of Eq. (2.2 that could be written down in a simple form, such as $y(x) = \sin(x)$ etc.
A 2nd order homogeneous linear differential equation for the function $y(x)$ has the form

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0, \tag{2.3}$$

i.e. the term $f(x)$ is zero on the r.h.s. of Eq.(2.2).
A 2nd order inhomogeneous linear differential equation for the function $y(x)$ **with constant coefficients** has the form

$$y''(x) + py'(x) + qy(x) = f(x), \tag{2.4}$$

where p and q are real numbers, $f(x)$ is a known function of x, and $y(x)$ is the function one would like to calculate.
A 2nd order homogeneous linear differential equation for the function $y(x)$ **with constant coefficients** has the form

$$y''(x) + py'(x) + qy(x) = 0, \tag{2.5}$$

where p and q are real numbers, and $y(x)$ is the function one would like to calculate.

Initial Value Problem for 2nd order differential equation for a function $y(x)$: To solve the initial value problem for a 2nd order differential equation for a function $y(x)$ means to solve $y(x)$ for the specific, given **initial conditions**

$$y(x = x_0) = y_0, \quad y'(x = x_0) = y'_0. \tag{2.6}$$

In the example of our harmonic oscillator this means that we start the motion at $t = t_0 = 0$ at the initial position $x(t_0) = x_0$ with the initial velocity $\dot{x}(t_0) = \dot{x}_0$.

2.1.3 How to Solve Them

In general, there is no recipe or general method of how to solve a given differential equation. In this lecture, we only discuss the 2nd order inhomogeneous linear differential equation for the function $y(x)$ with constant coefficients, for which there is a general method. 'Differential Equations' is a difficult topic, and still today a research subject in mathematics. Generations of people have tried to solve differential equations by finding new exact solutions, developing approximation techniques etc. For example, a big problem in Einstein's theory of gravitation is that the fundamental (partial) differential equations are known, but only very few exact solutions are known. This is still a hot topic today.

To warm up a bit, we solve a few simple cases of Eq.(2.1).

EXAMPLE: a particle of mass m under a constant external force $f_e(x) = f_e$ that does not depend on x. We have

$$\begin{aligned} \ddot{x}(t) &= \frac{1}{m} f_e \rightsquigarrow \dot{x}(t) = \frac{1}{m} f_e t + \dot{x}(0) \rightsquigarrow \\ x(t) &= \frac{1}{2m} f_e t^2 + \dot{x}(0) t + x(0). \end{aligned} \tag{2.7}$$

Here, the values $x(0)$ and $\dot{x}(t = 0)$ determine the initial condition at $t = 0$.
CHECK: go back to Pisa (Galilei) and establish the relation between this equation and the experiment of a freely falling mass m. In a 'Gedankenexperiment' (thought experiment), change the initial conditions $\dot{x}(t = 0)$ and $x(0)$ and discuss what changes then. What does a positive or a negative f_e mean?

2.2 2nd order homogeneous linear differential equations with constant coefficients I

We recall that this type of equation has the form

$$y''(x) + py'(x) + qy(x) = 0, \tag{2.8}$$

where p and q are real numbers, and $y(x)$ is the function one would like to calculate. An example is the differential equation of the damped linear harmonic oscillator

$$\ddot{x}(t) + \frac{\gamma}{m}\dot{x}(t) + \frac{k}{m}x(t) \quad = \quad 0, \quad k > 0, \gamma > 0, \tag{2.9}$$

cf. Eq.(2.1).

2.2.1 $y''(x) + qy(x) = 0, q > 0$

This is the case $p = 0$ of Eq. (2.8). An example for this is the differential equation of the undamped linear harmonic oscillator

$$\ddot{x}(t) + \frac{k}{m}x(t) \quad = \quad 0, \tag{2.10}$$

where $k > 0$ here, cf. Eq.(2.1). From our physical intuition, we know that the mass point described by Eq.(2.10) performs oscillations at an **angular frequency** ω. Therefore, we try sin and cos functions as solution: If we write

$$\begin{aligned} x(t) &= x_1 \sin(\omega t) \rightsquigarrow \dot{x}(t) = x_1\omega\cos(\omega t) \\ \rightsquigarrow \ddot{x}(t) &= -x_1\omega^2\sin(\omega t) = -\omega^2 x(t). \end{aligned} \tag{2.11}$$

Here, x_1 is an arbitrary constant. The function $x(t) = x_1\sin(\omega t)$ fulfills the differential equation Eq. (2.10), if

$$\omega^2 = \frac{k}{m}. \tag{2.12}$$

If on the other hand we write

$$\begin{aligned} x(t) &= x_2 \cos(\omega t) \rightsquigarrow \dot{x}(t) = -x_2\omega\sin(\omega t) \\ \rightsquigarrow \ddot{x}(t) &= -x_2\omega^2\cos(\omega t) = -\omega^2 x(t), \end{aligned} \tag{2.13}$$

we again recognise that the function$x(t) = x_2\cos(\omega t)$ fulfills the differential equation Eq. (2.10), if $\omega^2 = k/m$ (same as before). Again, x_2 is an arbitrary constant. Therefore, we find **two solutions of the second order differential equation** Eq. (2.10). Now we are a bit confused. Let us summarize what we have found so far, using our 'mathematical notation',

$$\begin{aligned} y''(x) + qy(x) &= 0, \quad q > 0 \rightsquigarrow \\ y(x) &= y_1(x) = y_1\sin(\sqrt{q}x), \quad y(x) = y_2(x) = y_2\cos(\sqrt{q}x). \end{aligned} \tag{2.14}$$

We now make an important observation:

THEOREM: With two solutions $y_1(x)$ and $y_2(x)$ of a linear homogeneous differential equation, also the sum $y_1(x) + y_2(x)$ is a solution of the linear homogeneous differential equation.
PROOF:

$$\begin{aligned} y_1''(x) + p(x)y_1'(x) + q(x)y_1(x) = 0, y_2''(x) + p(x)y_2'(x) + q(x)y_2(x) &= 0 \rightsquigarrow \\ \left[y_1''(x) + y_2''(x)\right] + p(x)\left[y_1'(x) + y_2'(x)\right] + q(x)\left[y_1(x) + y_2(x)\right] &= 0 \rightsquigarrow \\ \left[y_1 + y_2\right]''(x) + p(x)\left[y_1 + y_2\right]'(x) + q(x)\left[y_1(x) + y_2(x)\right] &= 0. \end{aligned}$$

We have used the fact that the sum of the derivatives of two functions is the derivative of the sum of the functions.

The **general solution** of $y''(x) + qy(x) = 0, q > 0$ can be written as the sum

$$y''(x) + qy(x) = 0 \rightsquigarrow y(x) = y_1 \sin(\sqrt{q}x) + y_2 \cos(\sqrt{q}x). \tag{2.15}$$

2.2.2 Initial Value Problem for $\ddot{x}(t) + \omega^2 x(t) = 0$

This is the equation of the undamped linear harmonic oscillator. Note that we write $x(t)$ instead of $y(x)$ here. We have found the general solution as

$$x(t) = x_1 \sin(\omega t) + x_2 \cos(\omega t), \tag{2.16}$$

where $x(t)$ is the position x at time t. As mentioned above, in order to know $x(t)$ at all times later than, say, $t = 0$, we must specify the **initial conditions**, i.e. the initial position of the particle $x_0 = x(t = 0)$ and its initial velocity $v_0 = \dot{x}(t = 0)$, i.e.

$$\begin{aligned} x_0 = x(t = 0) &= x_1 \sin(\omega 0) + x_2 \cos(\omega 0) = x_2 \\ v_0 = \dot{x}(t = 0) &= x_1\omega \cos(\omega t) - x_2\omega \sin(\omega t)|_{t=0} = x_1\omega. \end{aligned} \tag{2.17}$$

Therefore, we can express the parameters x_1 and x_2 by the given initial values x_0 and v_0 and obtain

$$x(t) = \frac{v_0}{\omega} \sin(\omega t) + x_0 \cos(\omega t). \tag{2.18}$$

2.2.3 $y''(x) + qy(x) = 0, q < 0$

We notice that for $q < 0$ the argument in the sin and cos in Eq.(2.15) becomes imaginary since $\sqrt{q} = \sqrt{-|q|} = i\sqrt{|q|}$ for $q < 0$. Let us find a solution by recalling that the exponential function $f(x) = \exp(x)$ fulfills

$$f(x) = e^x \rightsquigarrow f'(x) = e^x \rightsquigarrow f''(x) = e^x \rightsquigarrow ... \tag{2.19}$$

More generally, we have

$$\begin{aligned} f(x) &= e^{\lambda x} \rightsquigarrow f'(x) = \lambda e^{\lambda x} \rightsquigarrow f''(x) = \lambda^2 e^{\lambda x} \rightsquigarrow f''(x) = \lambda^2 f(x) \qquad (2.20) \\ f(x) &= e^{-\lambda x} \rightsquigarrow f'(x) = -\lambda e^{-\lambda x} \rightsquigarrow f''(x) = (-\lambda)^2 e^{-\lambda x} \rightsquigarrow f''(x) = \lambda^2 f(x). \end{aligned}$$

Comparing this to our differential equation,

$$y''(x) - |q|y(x) = 0 \Leftrightarrow y''(x) = |q|y(x), \qquad (2.21)$$

we recognize by comparing with Eq. (2.20) that two independent solutions of Eq. (2.21) are

$$\begin{aligned} y''(x) - |q|y(x) &= 0, \quad q \neq 0 \rightsquigarrow \\ y_1(x) &= y_1 e^{\sqrt{|q|}x}, \quad y_2(x) = y_2 e^{-\sqrt{|q|}x}. \end{aligned} \qquad (2.22)$$

As above, the most general solution again is the sum of these two, i.e. the **linear combination** of $e^{-\sqrt{|q|}x}$ and $e^{\sqrt{|q|}x}$ with the two independent constants y_1 and y_2,

$$\begin{aligned} y''(x) - |q|y(x) &= 0 \rightsquigarrow \\ y(x) &= y_1 e^{\sqrt{|q|}x} + y_2 e^{-\sqrt{|q|}x}. \end{aligned} \qquad (2.23)$$

2.2.4 $y''(x) + qy(x) = 0$, summary

We summarize the two pairs of solutions for $q > 0$ and $q = -|q| < 0$ of $y''(x)+qy(x) = 0$ in a table:

$y''(x) + qy(x) = 0, \quad q > 0$	$y''(x) + qy(x) = 0, \quad q = -\lvert q\rvert < 0$
two solutions $y_1(x) = y_1 \sin(\sqrt{q}x)$, $y_2(x) = y_2 \cos(\sqrt{q}x)$	two solutions $y_1(x) = y_1 e^{\sqrt{\lvert q\rvert}x}$, $y_2(x) = y_2 e^{-\sqrt{\lvert q\rvert}x}$
general solution $y(x) = y_1 \sin(\sqrt{q}x) + y_2 \cos(\sqrt{q}x)$	general solution $y(x) = y_1 e^{\sqrt{\lvert q\rvert}x} + y_2 e^{-\sqrt{\lvert q\rvert}x}$
character: oscillatory (sin and cos)	character: exponential (decreasing and incr.)

Note that the sign of q makes all the difference!

2.3 2nd order homogeneous linear differential equations with constant coefficients II

Now we attack the case of arbitrary p and q in our differential equation

$$y''(x) + py'(x) + qy(x) = 0. \qquad (2.24)$$

Remember that for $p > 0$ and $q > 0$ this corresponds to the differential equation Eq. (2.1) of the damped linear harmonic oscillator. We already know that this system performs oscillations ($\rightsquigarrow \sin, \cos$) that can be exponentially damped ($\rightsquigarrow \exp$). Therefore, we expect something related to $\sin, \cos, \exp$ functions. But these are all related to each other if we recall what we have learned about complex numbers:

$$\begin{aligned} \exp(ix) &= \cos(x) + i\sin(x), xreal \\ \cos(x) &= \frac{e^{ix} + e^{-ix}}{2} = Re[e^{ix}] \\ \sin(x) &= \frac{e^{ix} - e^{-ix}}{2i} = Im[e^{ix}]. \end{aligned} \tag{2.25}$$

Furthermore, for arbitrary complex $z = x + iy$,

$$e^z = e^{x+iy} = e^x e^{iy} = e^x[\cos(y) + i\sin(y)] = e^x \cos(y) + ie^x \sin(y). \tag{2.26}$$

The function e^z with complex z comprises the real exponential as well as sin and cos.

Let us therefore try an **exponential Ansatz** in Eq. (2.24),

$$y(x) = e^{zx} \rightsquigarrow y''(x) + py'(x) + qy(x) = [z^2 + pz + q]e^{zx} = 0. \tag{2.27}$$

We recognize that $y(x) = e^{zx}$ fulfills the differential equation, if the bracket [...] is zero:

$$[z^2 + pz + q] = 0. \tag{2.28}$$

This is a quadratic equation which in general has two solutions,

$$z^2 + pz + q = 0 \rightsquigarrow z_{1/2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}. \tag{2.29}$$

2.3.1 Case $\frac{p^2}{4} - q > 0$

In this case,

$$z_{1/2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q} \tag{2.30}$$

are both real and the two solutions fulfilling Eq. (2.24) are

$$\frac{p^2}{4} - q > 0 \rightsquigarrow y_1(x) = y_1 e^{[-\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}]x}, \quad y_2(x) = y_2 e^{[-\frac{p}{2} - \sqrt{\frac{p^2}{4} - q}]x} \tag{2.31}$$

The general solution is the linear combination of the two,

$$\frac{p^2}{4} - q > 0 \rightsquigarrow y(x) = y_1 e^{[-\frac{p}{2} + \sqrt{\frac{p^2}{4} - q}]x} + y_2 e^{[-\frac{p}{2} - \sqrt{\frac{p^2}{4} - q}]x}. \tag{2.32}$$

In this case there are no oscillations at all. The ‘damping term’ $py'(x)$ is too strong.

2.3.2 Case $\frac{p^2}{4} - q < 0$

In this case, the two zeros become complex:

$$z_{1/2} = -\frac{p}{2} \pm \sqrt{-\left|\frac{p^2}{4} - q\right|} = -\frac{p}{2} \pm i\sqrt{\left|\frac{p^2}{4} - q\right|} =: -\frac{p}{2} \pm i\Omega, \tag{2.33}$$

where we define an angular frequency $\Omega = \sqrt{|p^2/4 - q|}$. Now, the two solutions fulfilling Eq. (2.24) are

$$\frac{p^2}{4} - q < 0 \rightsquigarrow y_1(x) = y_1 e^{[-\frac{p}{2} + i\Omega]x}, \quad y_2(x) = y_2 e^{[-\frac{p}{2} - i\Omega]x}, \quad \Omega := \sqrt{\left|\frac{p^2}{4} - q\right|}. \tag{2.34}$$

The general solution is the linear combination of the two,

$$\frac{p^2}{4} - q < 0 \rightsquigarrow y(x) = y_1 e^{[-\frac{p}{2} + i\Omega]x} + y_2 e^{[-\frac{p}{2} - i\Omega]x}, \quad \Omega := \sqrt{\left|\frac{p^2}{4} - q\right|}. \tag{2.35}$$

We re–write this as

$$\begin{aligned} y(x) &= y_1 e^{[-\frac{p}{2} + i\Omega]x} + y_2 e^{[-\frac{p}{2} - i\Omega]x} = e^{-px}\left\{y_1 e^{i\Omega x} + y_2 e^{-i\Omega x}\right\} \\ &= e^{-px}\left\{y_1[\cos(\Omega x) + i\sin(\Omega x)] + y_2[\cos(\Omega x) - i\sin(\Omega x)]\right\} \\ &= e^{-px}\left\{[y_1 + y_2]\cos(\Omega x) + i[y_1 - y_2]\sin(\Omega x)\right\}. \end{aligned} \tag{2.36}$$

Now, this seems a bit odd since we have got a complex solution due to the term $i(y_1 - y_2)$. However, the constant coefficients y_1 and y_2 can be complex anyway (and still $y(x)$ is a solution of the differential equation). If we are only interested in real functions $y(x)$, we can re–define new constants $c_1 := y_1 + y_2$ and $c_2 := i[y_1 - y_2]$ such that the general solution becomes

$$\begin{aligned} y''(x) + py'(x) + qy(x) &= 0, \quad \frac{p^2}{4} - q < 0 \rightsquigarrow \\ y(x) &= e^{-px}\left\{c_1\cos(\Omega x) + c_2\sin(\Omega x)\right\}. \end{aligned} \tag{2.37}$$

Still c_1 and c_2 could be complex numbers, but we can choose them real if we only want real functions $y(x)$.

2.3.3 The Marginal Case $\frac{p^2}{4} - q = 0$

2.4 Inhomogeneous Equations

Now we arrive at the most general case we treat here, the **second order inhomogeneous linear differential equation for the function** $y(x)$ **with constant**

coefficients

$$y''(x) + py'(x) + qy(x) = f(x), \tag{2.38}$$

where p and q are real numbers, $f(x)$ is a known function of x, and $y(x)$ is the function one would like to calculate. In the following, we become a bit more 'physical' and discuss the differential equation of the forced, damped linear harmonic oscillator, Eq. (2.1),

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega^2 x(t) = \frac{1}{m} f(x), \quad \gamma > 0. \tag{2.39}$$

instead of Eq. (2.38). Since this means that $p > 0$ and $q > 0$ in Eq. (2.38), we are not that general, but the results shown here can be trasfered to the general case, too.

2.4.1 Initial Conditions for the Homogeneous Case

The solution for the homogeneous equation $f \equiv 0$ was obtained above,

$$\begin{aligned} y_h(x) &= e^{-\frac{p}{2}x} \left\{ c_1 \cos(\Omega x) + c_2 \sin(\Omega x) \right\}, \quad Eq.(2.38) \\ x_h(t) &= e^{-\gamma t} \left\{ x_1 \cos(\omega t) + x_2 \sin(\omega t) \right\}, \quad Eq.(2.39). \end{aligned} \tag{2.40}$$

Specifying to the **initial conditions**

$$x_h(t=0) = x_0, \quad \dot{x}_h(t=0) = v_0, \tag{2.41}$$

we find

$$x_h(t) = x_0 \left\{ e^{-\gamma t} \cos(\omega t) + \frac{\delta}{\omega} e^{-\gamma t} \sin(\omega t) \right\} + v_0 e^{-\gamma t} \frac{\sin(\omega t)}{\omega}. \tag{2.42}$$

$x_h(t)$ describes the motion of the harmonic oscillator for $f \equiv 0$ (homogeneous case). If we choose the initial time $t = t_0$ instead of $t = 0$, we have

$$\begin{aligned} x_h(t) &= x_0 \left\{ e^{-\gamma[t-t_0]} \cos(\omega[t-t_0]) + \frac{\delta}{\omega} e^{-\gamma[t-t_0]} \sin(\omega[t-t_0]) \right\} \\ &+ v_0 e^{-\gamma[t-t_0]} \frac{\sin(\omega[t-t_0])}{\omega}, \end{aligned} \tag{2.43}$$

i.e. everything remains the same; only the 'origin' of time t_0 is shifted, i.e the time scale is shifted by t_0.

EXERCISE: check that Eq. (2.42) fulfills the correct initial conditions!

2.4.2 The Inhomogeneous Case: Effect of the External Force

Now let us discuss the additional effect of the external force, i.e. the inhomogeneous term $f(t)/m$ in Eq. (2.39). First of all, we recognize that $f(t)/m$ is an additional acceleration, $a(t) = f(t)/m$, of the mass m due to the force $f(t)$ (**NEWTON !**). What is the additional displacement, $\Delta x(t)$, of the mass due to that acceleration? In a very short time interval from time $t = t'$ to $t = t' + \delta t'$, due to the acceleration $a(t')$ the mass aquires the additional velocity

$$v(t') = a(t')\delta t' = \frac{f(t')}{m}\delta t'. \tag{2.44}$$

The subsequent additional displacement $\Delta x(t > t')$ has to be proportional to that additional velocity and can be calculated using Eq.(2.43) with 'initial' additional shift $x_0 = 0$ and 'initial' additional velocity $v_0 = v(t')$,

$$\begin{aligned}
\Delta x(t > t') &= e^{-\gamma[t-t']}\frac{\sin(\omega[t-t'])}{\omega} \times v(t') \\
&= e^{-\gamma[t-t']}\frac{\sin(\omega[t-t'])}{\omega} \times \frac{f(t')}{m}\delta t', \\
&=: G(t-t') \times \frac{f(t')}{m}\delta t',
\end{aligned} \tag{2.45}$$

where in the last line we introduced an abbreviation for the term $e^{-\gamma[t-t']}\sin(\omega[t-t'])/\omega$. The function $G(t-t')$ is called **response function** (**Green's function**) of the harmonic oscillator since it describes its response to an additional, infinitesimal acceleration $f(t')\delta t'/m$. Note that we have made no additional assumptions on how this force $f(t')$ actually behaves as a function of time.

The total additional shift $x_f(t)$ at time t can be calculated from Eq.(2.45) by integrating the contributions from all times t' with $t_0 < t' < t$,

$$x_f(t) = \int_{t_0}^{t} dt' \Delta x(t > t') = \int_{t_0}^{t} dt' G(t-t')\frac{f(t')}{m}. \tag{2.46}$$

The position $x(t)$ at time x now is given by the contribution $x_h(t)$ (force $f = 0$) plus the additional shift $x_f(t)$ (force $f \neq 0$),

$$x(t) = x_h(t) + x_f(t) = x_h(t) + \int_{t_0}^{t} dt' G(t-t')\frac{f(t')}{m}. \tag{2.47}$$

Putting everything together, we find a somewhat lengthy, but very convincing expression (we set the initial time $t_0 = 0$ for simplicity),

$$
\begin{aligned}
x(t) &= x_0 \left\{ e^{-\gamma t} \cos(\omega t) + \frac{\delta}{\omega} e^{-\gamma t} \sin(\omega t) \right\} + v_0 e^{-\gamma t} \frac{\sin(\omega t)}{\omega} \\
&+ \int_0^t dt' e^{-\gamma[t-t']} \frac{\sin(\omega[t-t'])}{\omega} \frac{f(t')}{m}.
\end{aligned}
\tag{2.48}
$$