3. SERIES AND LIMITS

3.1 Finite and Infinite Series

3.1.1 Finite Series of Natural Numbers and Little Gauß, the Genius

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}. \tag{3.1}$$

First Proof (C. F. Gauß) for $n = 100$:

$$\sum_{k=1}^{100} k = (1 + 100) + (2 + 99) + (3 + 98) + ... + (50 + 51) = 50 \cdot 101 = 5050. \tag{3.2}$$

General proof by **induction**:
1. Induction Start ($n = 1$):

$$\sum_{k=1}^{1} k = 1 = \frac{1(1+1)}{2} \rightsquigarrow OK. \tag{3.3}$$

2. Induction Step ($n \to n+1$): Assume Eq.(3.1) is true for n, then show that it is true for $n+1$:

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2} \rightsquigarrow OK. \tag{3.4}$$

3.1.2 Finite Geometric Progression

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}, \quad x \neq 0. \tag{3.5}$$

Proof: Write

$$\begin{aligned} S_n = \sum_{k=0}^{n} x^k &= 1 + x + x^2 + ... + x^n \\ &= 1 + x(1 + x + x^2 + ... + x^{n-1} + x^n) - x^{n+1} \\ &= 1 + xS_n - x^{n+1} \rightsquigarrow \\ S_n &= \frac{1-x^{n+1}}{1-x}. \end{aligned} \tag{3.6}$$

Alternative proof by **induction**: home exercise.

3.1.3 Binomial

$$(x+y)^n = \sum_{k=0}^{n} \left(\begin{array}{c} \mathrm{n} \\ \mathrm{k} \end{array} \right) x^{n-k} y^k. \tag{3.7}$$

Here, we defined the **binomial coefficient**

$$\left(\begin{array}{c} \mathrm{n} \\ \mathrm{k} \end{array} \right) = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdot ...(n-k+1)}{1\cdot 2\cdot ...\cdot k}. \tag{3.8}$$

The proof of Eq.(**??**) goes again via induction $n \to n+1$. Not shown here. Examples:

$$\begin{aligned} (x+y)^2 &= x^2 + 2xy + y^3 \\ (x+y)^3 &= x^3 + 3x^2 y + 3xy^2 + y^3. \end{aligned} \tag{3.9}$$

3.1.4 Infinite Series

Definition

A series

$$S := \sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + ... = \lim_{n\to\infty} S_n, \quad S_n := \sum_{k=0}^{n} a_k \tag{3.10}$$

is called **infinite series**. It is the limit of the sequence of finite series S_n where the upper limit n tends toward infinity. In contrast to the finite series S_n, the infinite series S can **diverge**. S is said to be **convergent** is S_n approaches a finite limit as $n \to \infty$.

Example: constant a_k

$$a_k = 1 \rightsquigarrow \sum_{k=0}^{\infty} a_k = 1 + 1 + 1 + 1 + ... \tag{3.11}$$

is divergent because the result is not a finite number.

Example: geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad |x| < 1. \tag{3.12}$$

LEARN THIS ONE BY HEART. This series converges for arbitrary (real or complex) numbers x with $|x| < 1$.
EXERCISE: Sketch the condition $|z| < 1$ for complex z as an an area in the complex plane.
Proof of Eq.(3.33):

$$\begin{aligned} S &= \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + ... = 1 + x(1 + x + x^2 + ...) = 1 + x \cdot S \rightsquigarrow \\ S &= \frac{1}{1-x}. \end{aligned} \tag{3.13}$$

The problem with infinite series is that often it is not easy to decide if or if not they converge, e.g. for which values of x in the above example.
NECESSARY for convergence of $S = \sum_{k=0}^{\infty} a_k$ is that $a_k \to 0$ as $k \to \infty$.
SUFFICIENT for convergence: **ratio test** with positive result:

Ratio test: Consider the series $S := \sum_{k=0}^{\infty} a_k$ and assume $a_k \neq 0$ for all $k > k_0$. Define the ratio

$$\begin{aligned} R &:= \lim_{k \to infty} \left| \frac{a_{k+1}}{a_k} \right| \rightsquigarrow \\ &\text{R¡1 series is convergent} \\ &\text{R¿1 series is divergent.} \end{aligned} \tag{3.14}$$

For $R = 1$, the ratio test can't decide whether the series is convergent or divergent.

3.2 Taylor–Series

One of the main motivations to investigate infinite series is the desired to write arbitrary functions $f(x)$ as polynomials of infinite degree, i.e.

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^4 + ... \\ &= \sum_{k=0}^{\infty} a_k x^k. \end{aligned} \tag{3.15}$$

For each fixed x, this is an infinite series of the form $S := \sum_{k=0}^{\infty} b_k$ with $b_k := a_k x^k$. An important question is, e.g., how to determine the coefficients a_k for a given function $f(x)$, and to decide for which values of x the series for $f(x)$ does converge.

3.2.1 The Exponential Function

We already know one example for such a series which is the exponential function

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{3.16}$$

You have to remember this formula throughout your whole life. This series converges for arbitrary values of (complex or real) x since (ratio test!)

$$R := \lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = lim_{n\to\infty} \frac{x}{n+1} = 0. \tag{3.17}$$

By use of the this **exponential series** one defines the famous **Euler number**

$$e := \sum_{n=0}^{\infty} \frac{1}{n!} = \exp(1). \tag{3.18}$$

3.2.2 Power Series for $\sin(x)$ and $\cos(x)$

We repeat our result for the series that define sin and cos:

$$\begin{aligned} \sin(x) &:= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \\ \cos(x) &:= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... \end{aligned} \tag{3.19}$$

3.2.3 General Case

Now we treat the case of an arbitrary function

$$f(x) \quad = \quad a_0 + a_1 x + a_2 x^2 + a_3 x^4 + ... = \sum_{k=0}^{\infty} a_k x^k. \tag{3.20}$$

The above equation means that we try to represent the function by an 'infinite' polynomial. In the following, we assume that all derivatives of $f(x)$, i.e. $f'(x) =: f^{(1)}(x)$, $f''(x) =: f^{(2)}(x)$, $f'''(x) =: f^{(3)}(x)$, ... etc. exist. We write

$$\begin{aligned}
f(x=0) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...\big|_{x=0} = a_0 \\
f'(x=0) &= a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + ...\big|_{x=0} = a_1 \\
f''(x=0) &= 2a_2 + 2\cdot 3a_3 x + 3\cdot 4a_4 x^2 + ...\big|_{x=0} = 1\cdot 2a_2 \\
f^{(3)}(x=0) &= 2\cdot 3a_3 + 2\cdot 3\cdot 4a_4 x + ...|_{x=0} = 1\cdot 2\cdot 3a_3 \\
... &= ... \\
f^{(n)}(x=0) &= 1\cdot 2\cdot ...\cdot n\cdot a_n = n! a_n \\
\rightsquigarrow a_n &= \frac{f^{(n)}(x=0)}{n!}.
\end{aligned} \tag{3.21}$$

Collecting all terms, we find the

Taylor expansion of $f(x)$ around $x = 0$,

$$f(x) = \frac{f(x=0)}{0!} + \frac{f'(x=0)}{1!}x + \frac{f''(x=0)}{2!}x^2 + ... = \sum_{n=0}^{\infty} \frac{f^{(n)}(x=0)}{n!}x^n. \tag{3.22}$$

We define the truncated Taylor series

$$f_N(x) := \sum_{n=0}^{N} \frac{f^{(n)}(x=0)}{n!}x^n \rightsquigarrow f(x) = \lim_{N\to\infty} f_N(x). \tag{3.23}$$

The truncated Taylor series for finite N is used as an approximation for the function $f(x)$, i.e. the infinite Taylor series. For larger and larger N, we expect that this approximation of the function $f(x)$ by a polynomial of degree N becomes better and better. Let us look at an example how this works:

3.2.4 Example: The Exponential Function $\exp(x)$

We calculate the Taylor series of $f(x) = \exp(x)$ around $x = 0$. To do so, we have to calculate the derivatives

$$\begin{aligned} f^{(0)}(x=0) \equiv f(x=0) &= e^x|_{x=0} = 1 \\ f^{(1)}(x=0) &= e^x|_{x=0} = 1 \\ f^{(2)}(x=0) &= e^x|_{x=0} = 1 \\ \dots & \quad \dots \end{aligned} \tag{3.24}$$

This is particularily simple because all the derivatives of e^x are e^x. This means that

$$\begin{aligned} f(x) &= \lim_{N\to\infty} f_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{f^{(n)}(x=0)}{n!} x^n \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \end{aligned} \tag{3.25}$$

We recognise that the Taylor expansion of $f(x) = \exp(x)$ just reproduces our old defining Eq. (3.16).

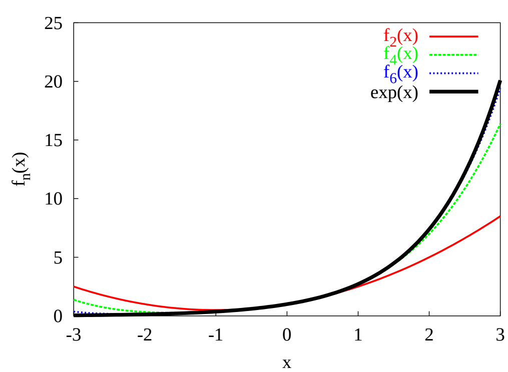

Fig. 3.1:
Approximation of the function $f(x) = \exp(x)$ by the truncated Taylor Series $f_N(x)$, Eq. (3.25), for $N = 2, 4, 6$. For the interval $x \in [-3, 3]$ shown here, the approximation of $\exp(x)$ by $f_6(x)$ is already very good.

3.3 Taylor–Expansion of Functions

3.3.1 Convergence: Expansion of $f(x) = \ln(1+x)$

The derivatives of this function are

$$\begin{aligned}
f(x) &= \ln(1+x) \rightsquigarrow f(0) = \ln(1) = 0 \\
f'(x) &= (1+x)^{-1} \\
f''(x) &= (-1)(1+x)^{-2} \\
f^{(3)}(x) &= 2(1+x)^{-3} \\
f^{(4)}(x) &= -6(1+x)^{-4} \\
\ldots & \quad \ldots \\
f^{(n)}(x) &= (-1)^{n+1}(n-1)!(1+x)^{-n} \rightsquigarrow f^{(n)}(x=0) = (-1)^{n+1}(n-1)! \qquad (3.26)
\end{aligned}$$

We use this to expand $f(x)$ around $x = 0$,

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x=0)}{n!} x^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n (n-1)!}{n!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}. \qquad (3.27)$$

Now we ask: for which values of x does this Taylor series actually converge? We use the ratio test and write

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} &= \sum_{n=1}^{\infty} a_n \rightsquigarrow a_n = \frac{(-1)^{n+1} x^n}{n} \\
R &:= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n}{n+1} |x| = |x| . \qquad (3.28)
\end{aligned}$$

From Eq. (3.14), we recognise that the series

- converges for $|x| < 1$.
- diverges for $|x| > 1$.

At $x = -1$ we don't expect the series to converge because $\ln(1 + (-1)) = \ln(0)$ is undefined (minus infinity). To decide what happens at $x = 1$, we have to invoke an additional convergence test:

Leibnitz' test for alternating series: The alternating series

$$\sum_{k=0}^{\infty} (-1)^k |a_k| \qquad (3.29)$$

converges, if $|a_{k+1}| < |a_k|$ for all k, and $\lim_{k\to\infty} a_k = 0$.

We apply this rule to the case $x = 1$ of our series Eq. (**??**) for $\ln(1+x)$: At $x = 1$, $|a_{n+1}| = 1/(n+1) < |a_n| = 1/n$ and $\lim_{n\to\infty} |a_n| = 0$, that means the Leibnitz' test tells us that the series converges at $x = 1$. The result gives us a famous formula for $\ln 2$,

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - ..., \tag{3.30}$$

and we summarise our results for $f(x) = \ln(1+x)$ as

$$\ln(1+x) == \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \quad |x| < 1. \tag{3.31}$$

We say that the **radius of convergence** R of this series is $R = 1$. Beyond that radius, the series diverges and does no longer represent the function $\ln(1+x)$. In other words, the Taylor series Eq. (3.31) is only useful for 'small' x.

3.3.2 A Feynman–like, clever trick to generate a Taylor Series is a different way

Let us write $\ln(1+x)$ in a 'complicated way', i.e. as an integral:

$$\ln(1+x) = \int_0^x \frac{dt}{1+t}. \tag{3.32}$$

Now, we use our result for the geometric series, Eq.(3.33),

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1. \tag{3.33}$$

(**'LEARN THIS ONE BY HEART'**) with $t = -x$, which leads to

$$\begin{aligned} \ln(1+x) &= \int_0^x dt \frac{1}{1+t} = \int_0^x dt \sum_{n=0}^{\infty} (-t)^n \\ &= \int_0^x dt(1 - t + t^2 - t^3 + ...) \end{aligned} \tag{3.34}$$

We integrate this term by term, which is easy,

$$\begin{aligned} \ln(1+x) &= \int_0^x dt(1 - t + t^2 - t^3 + ...) = \sum_{n=0}^{\infty} \int_0^x dt(-t)^n = x - \frac{x^2}{2} + \frac{x^3}{3} - ... = \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \end{aligned} \tag{3.35}$$

which is the same as Eq. (3.31)

3.3.3 Taylor expansion of $f(x)$ around an arbitrary $x = a$

So far we have always expanded our functions $f(x)$ in the vicinity of $x = 0$, i.e. 'around' $x = 0$:

Taylor expansion of $f(x)$ around $x = 0$,

$$f(x) = \frac{f(x=0)}{0!} + \frac{f'(x=0)}{1!}x + \frac{f''(x=0)}{2!}x^2 + ... = \sum_{n=0}^{\infty} \frac{f^{(n)}(x=0)}{n!}x^n. \quad (3.36)$$

The Taylor expansion of a function $f(x)$ near $x = a$ is performed in an analogous way, but with $x = 0$ replaced by $x = a$, and $x = x - 0$ replaced by $x - a$:

Taylor expansion of $f(x)$ around $x = a$,

$$f(x) = \frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n. \quad (3.37)$$

In some books, the special case of a Taylor series around $x = 0$ is called **Maclaurin Series** .

3.4 Further Examples for Series and Limits

3.4.1 Newtonian Limit of Relativistic Energy

According to Einstein, the total energy of a particle of rest mass m_0 and velocity v is

$$E = \frac{m_0 c^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}, \quad (3.38)$$

where c is the speed of light on vacuum. We would like to find an approximation of this formula for small velocities $v \ll c$, in order to compare to Newton's expression for the kinetic energy, $E_{\text{kin}} = (1/2)m_0 v^2$. Defining $\beta := v/c$, we recognise that our mathematical task is to $(x := \beta^2)$

Expand $f(x) = \frac{1}{\sqrt{1-x}}$ around $x = 0$:

We write $f(x) = (1-x)^{-1/2}$ and

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= (-1)(-1/2)(1-x)^{-3/2}\Big|_{x=0} = 1/2 \\ f''(0) &= (-1)(-1/2)(-1)(-3/2)(1-x)^{-3/2}\Big|_{x=0} = 3/4 \\ &\ldots \end{aligned} \tag{3.39}$$

With our

Taylor expansion of $f(x)$ around $x = 0$,

$$f(x) = \frac{f(x=0)}{0!} + \frac{f'(x=0)}{1!}x + \frac{f''(x=0)}{2!}x^2 + ... = \sum_{n=0}^{\infty} \frac{f^{(n)}(x=0)}{n!}x^n, \tag{3.40}$$

we find the first terms as

$$f(x) = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + ... \tag{3.41}$$

(note that $2! = 2$). Therefore, with $x = \beta^2 = (v/c)^2$, we obtain

$$E = m_0c^2 \left[1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 + O\left(\frac{v}{c}\right)^6\right]. \tag{3.42}$$

Here, we introduced the **O–symbol** (speak 'order of'), i.e. $O(x)^6$ means 'terms of order x^6 or higher powers' like x^7, x^8 etc. This is a convenient way to express that in a Taylor expansion with the first few terms written down as above, there are higher order terms to follow that one does not care to write down explicitely here. These higher order terms in fact become smaller and smaller for $|x| < 1$.

We can take use of the O–symbol to write

$$E = m_0c^2 + \frac{1}{2}m_0v^2 + O\left(\frac{v}{c}\right)^4. \tag{3.43}$$

This shows that the first term in the total energy is a velocity–independent rest energy of the particle, and the second term is the *lowest order approximation* to its kinetic energy. The relativistic correction to the kinetic energy if of order $(v/c)^4$, i.e. very small for velocities small compared to the speed of light.

3.4.2 Limits

Expressions of the type '0/0'

Often we have to discuss and sketch functions like

$$f(x) = \frac{\sin(x)}{x} \tag{3.44}$$

with a seemingly complicated behaviour at $x = 0$. It looks like an expression $0/0$, i.e. not well defined. However, a closer look shows that this is not the case: we expand the *sin* by its Taylor series near $x = 0$, i.e.

$$f(x) = \frac{\sin(x)}{x} = \frac{x - \frac{x^3}{3!} + O(x^5)}{x} = 1 - \frac{x^2}{3!} + O(x^4). \tag{3.45}$$

This means, that if x approaches $x = 0$ we have the finite value $f(x = 0) = 1$, i.e.

$$\lim_{x \to 0} f(x) = 1. \tag{3.46}$$

The deviation from $f(x = 0) = 1$ close to $x = 0$ is described by the second term, $-x^2/6$, i.e. a quadratic decrease of the function for small values of x. This kind of analysis helps a lot when sketching functions.