4. FUNCTIONS OF MORE THAN ONE VARIABLE

4.1 Function of several variables

Definition: A real function $f(x_1, ..., x_n)$ of n real variables x_1,...,x_n is a map

$$f : R^n \to R, \quad \mathbf{x} := (x_1, x_2, ..., x_n) \to f(\mathbf{x}) = f(x_1, ..., x_n). \tag{4.1}$$

In the following, we mainly discuss functions of two variables x_1 and x_2, i.e. the case $n = 2$. Such functions can be represented by a **three–dimensional surface plot**, where the value $z = f(x, y)$ at each point (x, y) in the x–y plane is plotted in the z–direction over the x–y plane. Here are two examples:

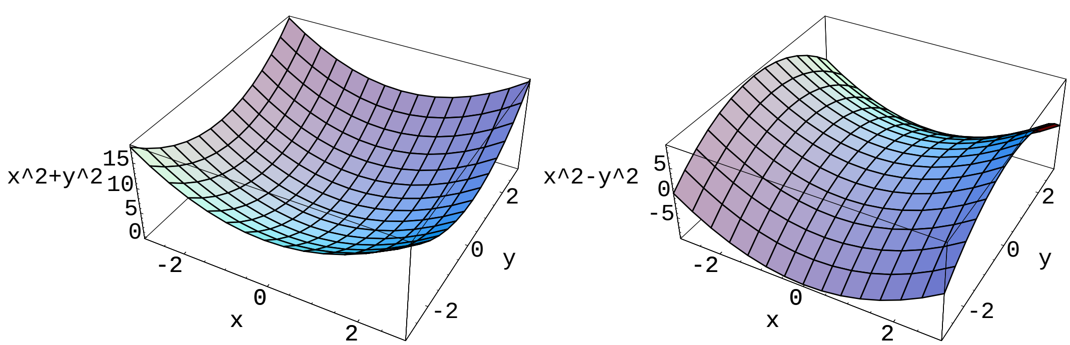

Fig. 4.1: Examples of functions $f(x, y)$ of two variables x and y: Paraboloid $f(x, y) = x^2+y^2$ (LEFT), saddle $f(x, y) = x^2 - y^2$ (RIGHT).

The Paraboloid $f(x, y) = x^2 + y^2$

We can understand this graph as follows:

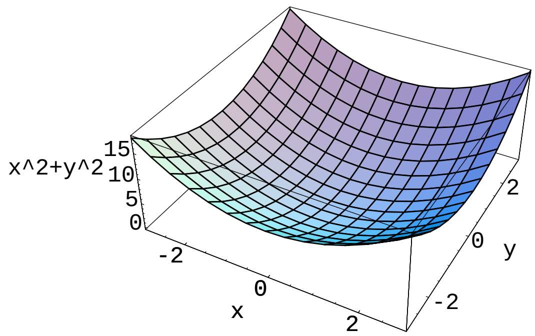

Fig. 4.2: Paraboloid $f(x, y) = x^2 + y^2$.

1. If we keep y fixed and change x, we have a parabola for each y, e.g.

$$\begin{aligned} f(x,0) &= x^2 \\ f(x,-1) &= x^2+1 \\ f(x,1) &= x^2+1 \\ f(x,2) &= x^2+4 \\ &\ldots \end{aligned}$$

If we keep x fixed and change y, we have a parabola for each x, e.g.

$$\begin{aligned} f(0,y) &= y^2 \\ f(-1,y) &= 1+y^2 \\ f(1,y) &= 1+y^2 \\ f(3,y) &= 9+y^2 \\ &\ldots \end{aligned}$$

These are cross–sections of the graph in x and y direction. It is important that you learn to visualise these cross–sections in your mind.

2. If we keep the value of the function fixed, i.e. $z = f(x, y) = z_0 = const > 0$, we find circles

$$z_0 = f(x, y) = x^2 + y^2. \tag{4.2}$$

Exercise: visualise these circles from the figure above. What are the radii of the circles for a given height $f(x, y) = z_0$?

The Saddle $f(x, y) = x^2 - y^2$

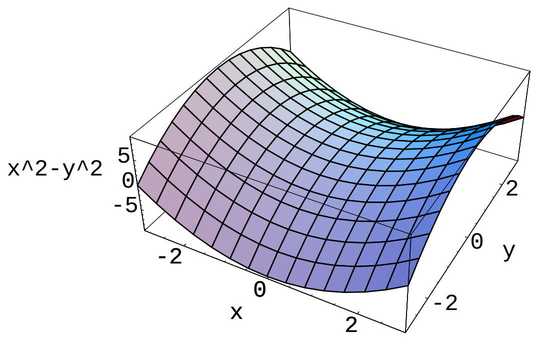

Fig. 4.3: Saddle $f(x, y) = x^2 - y^2$.

We can understand this graph as follows:
1. If we keep y fixed and change x, we have a parabola for each y, e.g.

$$\begin{aligned} f(x,0) &= x^2 \\ f(x,-1) &= x^2 - 1 \\ f(x,1) &= x^2 - 1 \\ f(x,2) &= x^2 - 4 \\ &\dots \end{aligned}$$

These are parabolas bending upwards but with their origin shifted to negativ values.

If we keep x fixed and change y, we have a parabola for each x, e.g.

$$\begin{aligned} f(0,y) &= -y^2 \\ f(-1,y) &= 1 - y^2 \\ f(1,y) &= 1 - y^2 \\ f(3,y) &= 9 - y^2 \\ &\dots \end{aligned}$$

These are parabolas bending downwards but with their origin shifted to positiv values. We can built up the whole graph from these cross-sections. The interesting thing is that we understand the **global** shape of the surface $f(x, y)$, i.e. its saddle–shape, only form ‘glueing’ together all the cross–sections.

The most interesting points in both the paraboloid and the saddle are the extrema at $(x, y) = (0, 0)$: for the paraboloid, this is a global minimum, for the saddle this is neither a minimum (it is a minimum in x–direction only) nor a maximum (it is a maximum in y–direction only): it is a **saddle–point**.

4.1.1 Symmetries

The Paraboloid $f(x, y) = x^2 + y^2$

We consider the circle of fixed radius r,

$$r^2 = x^2 + y^2 \tag{4.3}$$

in the x–y plane. For all points on this circle, the function $f(x, y) = x^2 + y^2$ has the same value $f(x, y) = r^2$. A rotation of a point (x, y) on this circle around the origin $(x, y) = (0, 0)$ does not alter $f(x, y)$. In fact, if we rotate $f(x, y)$ continuously around the z–axis, $f(x, y)$ remains invariant. The function $f(x, y)$ has a continous rotation symmetry. Therefore, it is also called 'rotational paraboloid' sometimes. We can build up the whole surface $f(x, y)$ from rings with radius r stacked on top of each other in the z–direction. Again, when dealing with functions of more than one variable, it is important that you develop this geometric kind of thinking.

The function $f(x, y)$ also has other symmetries: $f(x, y) = f(-x, y) = f(x, -y) = f(-x, -y)$ (reflection of one or two of its variables).

4.2 Partial Derivatives

4.2.1 Reminder: Derivative of a function of one variable

The derivative $f'(x)$ of a function $f(x)$ gives the slope of the function at x. It is defined as

$$\frac{df(x)}{dx} \equiv f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}. \tag{4.4}$$

Change of height: the quantity

$$f'(x)dx$$

gives the change of the height of the curve $f(x)$ (measured from the x–axis) at the point x, if we move a tiny step dx along the x–axis.

4.2.2 Derivatives for functions of two variables

For a function $f(x, y)$ with two independent variables, in a certain point (x, y) we can define the slope in either the x– or the y–direction. These two give rise to the **partial derivatives**

$$\begin{aligned} \frac{\partial}{\partial x} f(x, y) &:= \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \\ \frac{\partial}{\partial y} f(x, y) &:= \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}. \end{aligned} \tag{4.5}$$

Partial derivative $\frac{\partial}{\partial x} f(x, y_0)$

The geometrical meaning of this is a follows: we keep $y = y_0$ constant and consider the surface $f(x, y)$ along the x–direction, i.e. the curve $f(x, y_0)$ on the surface that appears through the cross–section with the plane $y = y_0$ parallel to the x–z–plane. The partial derivative $\frac{\partial}{\partial x} f(x, y_0)$ gives the slope of this curve at x. In other words: the partial derivative $\frac{\partial}{\partial x} f(x, y)$ gives the slope of the surface at (x, y) in x–direction. Change of height: the quantity

$$\frac{\partial}{\partial x} f(x, y) dx$$

gives the change of the height of the surface $f(x, y)$ (measured from the x–y–plane) at the point (x, y), if we move a tiny step dx along the x–direction.

Partial derivative $\frac{\partial}{\partial y} f(x_0, y)$

The geometrical meaning of this is a follows: we keep $x = x_0$ constant and consider the surface $f(x, y)$ along the y–direction, i.e. the curve $f(x_0, y)$ on the surface that appears through the cross–section with the plane $x = x_0$ parallel to the y–z–plane. The partial derivative $\frac{\partial}{\partial y} f(x_0, y)$ gives the slope of this curve at y. In other words: the partial derivative $\frac{\partial}{\partial y} f(x, y)$ gives the slope of the surface at (x, y) in y–direction. Change of height: the quantity

$$\frac{\partial}{\partial y} f(x, y) dy$$

gives the change of the height of the surface $f(x, y)$ (measured from the x–y–plane) at the point (x, y), if we move a tiny step dy along the y–direction.

Total change of height (total differential)

The quantity

$$df(x,y) := \frac{\partial}{\partial x} f(x,y)dx + \frac{\partial}{\partial y} f(x,y)dy \tag{4.6}$$

is called the **total differential of** $f(x,y)$ **at the point** (x,y) and gives the total change of the height of the surface $f(x,y)$ (measured from the x–y–plane) at the point (x,y), if we move a tiny step dx along the x–direction and a tiny step dy along the y–direction.

How to calculate partial derivatives

This is very simple:

- To calculate $\frac{\partial}{\partial x} f(x,y)$, we keep y fixed and differentiate $f(x,y)$ with respect to x. In doing so, y is regarded as a fixed parameter.
- To calculate $\frac{\partial}{\partial y} f(x,y)$, we keep x fixed and differentiate $f(x,y)$ with respect to y. In doing so, x is regarded as a fixed parameter.

Examples

$$\begin{aligned}
f(x,y) &= x^2 + y^2 \rightsquigarrow \frac{\partial}{\partial x} f(x,y) = 2x, \quad \frac{\partial}{\partial y} f(x,y) = 2y \\
f(x,y) &= x^2 y^3 \rightsquigarrow \frac{\partial}{\partial x} f(x,y) = 2xy^3, \quad \frac{\partial}{\partial y} f(x,y) = x^2 3y^2. \\
f(x,y) &= e^{-xy} \rightsquigarrow \frac{\partial}{\partial x} f(x,y) = -ye^{-xy}, \quad \frac{\partial}{\partial y} f(x,y) = -xe^{-xy}.
\end{aligned}$$

4.2.3 Higher Derivatives, Notation

Higher partial derivatives are easily defined: The second partial derivative $\frac{\partial^2}{\partial x^2} f(x,y)$ is the partial derivative with respect to x of the partial derivative $\frac{\partial}{\partial x} f(x,y)$, etc. To simplify the notation, one often defines

$$
\begin{aligned}
f_x \equiv \frac{\partial}{\partial x} f(x,y), \quad f_{xx} &\equiv \frac{\partial^2}{\partial x^2} f(x,y) := \frac{\partial}{\partial x}\frac{\partial}{\partial x} f(x,y)\\
f_y \equiv \frac{\partial}{\partial y} f(x,y), \quad f_{yy} &\equiv \frac{\partial^2}{\partial y^2} f(x,y) := \frac{\partial}{\partial y}\frac{\partial}{\partial y} f(x,y)\\
f_{xy} &\equiv \frac{\partial^2}{\partial x \partial y} f(x,y) := \frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y)\\
f_{yx} &\equiv \frac{\partial^2}{\partial y \partial x} f(x,y) := \frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x,y).
\end{aligned} \tag{4.7}
$$

Examples of Higher Partial Derivatives

$$
\begin{aligned}
f(x,y) &= x^2 + y^2 \rightsquigarrow f_x(x,y) = 2x, \quad f_y(x,y) = 2y\\
f_{xx}(x,y) &= f_{yy}(x,y) = 2, \quad f_{xy}(x,y) = f_{yx}(x,y) = 0.
\end{aligned}
$$

4.3 Curves on Surfaces

4.3.1 Curves in the x–y–plane

Definition: A curve in the x–y–plane is a map

$$R \to R^2, \quad t \to \mathbf{x}(t) := (x(t), y(t)) \tag{4.8}$$

which associates with each values of the parameter t ('time' t) a point $(x(t), y(t))$ in the x–y–plane.

Examples

1. The circle around the origin,

$$(x(t), y(t)) = (r\cos(t), r\sin(t)). \tag{4.9}$$

Check that $x(t)^2 + y(t)^2 = r^2$ for all t.
2. The curve line

$$(x(t), y(t)) = (t^2, t). \tag{4.10}$$

Sketch this!

4.3.2 Curves on Surfaces

Consider a function $f(x, y)$, i.e. a surface $z = f(x, y)$ above the x–y–plane. Consider a curve $(x(t), y(t))$ in the x–y–plane. This curve defines a corresponding curve

$$z(t) = f(x(t), y(t)) \tag{4.11}$$

on the surface. Example: for $f(x, y) = x^2 + y^2$ and $(x(t), y(t)) = (r\cos(t), r\sin(t))$, $z(t) = r^2$. The circle in the x–y–plane corresponds to a ring hovering at a distance r^2 above the plane, being part of the surface of the paraboloid $x^2 + y^2$. Sketch the corresponding picture (lecture)!

4.3.3 Change of height along a Curve

Reminder: Total change of height (total differential)

$$df(x, y) := \frac{\partial}{\partial x} f(x, y)dx + \frac{\partial}{\partial y} f(x, y)dy$$

is called the **total differential of** $f(x, y)$ **at the point** (x, y) and gives the total change of the height of the surface $f(x, y)$ (measured from the x–y–plane) at the point (x, y), if we move a tiny step dx along the x–direction and a tiny step dy along the y–direction.

From this, we can calculate the change of the height of the curve $z(t) = f(x(t), y(t))$:

$$\frac{dz(t)}{dt} = \frac{df(x(t), y(t))}{dt} = \frac{\partial}{\partial x} f(x, y)\frac{dx(t)}{dt} + \frac{\partial}{\partial y} f(x, y)\frac{dy(t)}{dt}. \tag{4.12}$$

This is a **chain rule**

$$\frac{df(x(t), y(t))}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{4.13}$$

Example: $f(x, y) = x^2 + y^2$ and $(x(t), y(t)) = (t^2, t)$

We have

$$\frac{dz(t)}{dt} = = \frac{\partial}{\partial x} f(x, y)\frac{dx(t)}{dt} + \frac{\partial}{\partial y} f(x, y)\frac{dy(t)}{dt} = 2x(t) \cdot 2t + 2y(t) \cdot 1 = 4t^3 + 2t.$$

We can check this by direct calculation, $z(t) = t^4 + t^2 \rightsquigarrow dz(t)/dt = 4t^3 + 2t$. The general formula, however, makes it clear that there a two contributions to the change of the curve $z(t)$ on the surface: 1. the 'geometric change' (partial derivatives f_x, f_y) of the surface. 2. the 'kinematic change', i.e. the time derivatives $dx(t)/dt$, $dy(t)/dt$ that determine the speed by which we sweep along the curve $z(t)$.

Example: $f(x,y) = x^2 - y^2$ and $(x(t), y(t)) = (\cos t, \sin t)$

$$\begin{aligned}\frac{dz(t)}{dt} &= = \frac{\partial}{\partial x} f(x,y) \frac{dx(t)}{dt} + \frac{\partial}{\partial y} f(x,y) \frac{dy(t)}{dt} = 2x(t) \cdot (-\sin(t)) - 2y(t) \cdot \cos(t) \\ &= -2\cos(t)\sin(t) - 2\sin(t)\cos(t) = -2\sin(2t).\end{aligned}$$

Direct check with $z(t) = \cos^2(t) - \sin^2(t) = \cos(2t)$.

4.4 The Gradient

4.4.1 Definition of the Gradient

Definition: Let $f(x,y)$ be a real function of two variables. The **gradient** grad$f$ of f in the point (x_0, y_0) in the x–y–plane is the two–component *vector* of the partial derivatives f_x and f_y of f,

$$\mathrm{grad} f(x_0, y_0) \equiv \nabla f(x_0, y_0) = (f_x(x_0, y_0), f_y(x_0, y_0)) \,. \tag{4.14}$$

The symbol ∇ is called 'Nabla'–operator. Note: the gradient of f in the point (x_0, y_0) is a two–dimensional vector in the x–y–plane attached to that point. The map $(x,y) \to \nabla f(x,y)$ defines a **vector field**, i.e. to each point (vector) (x,y) in the x–y–plane, a vector $\nabla f(x,y)$ is attached.

4.4.2 Examples

Paraboloid $f(x,y) = x^2 + y^2$

In this case,

$$\nabla f(x,y) = (2x, 2y).$$

Sketch this vector field in the x–y–plane (solution is given in the lecture).

Hyperboloid $f(x,y) = x^2 - y^2$

In this case,

$$\nabla f(x,y) = (2x, -2y).$$

Sketch this vector field in the x–y–plane (solution is given in the lecture).

4.4.3 Gradient and Differential; Geometrical Meaning

Reminder: Total change of height (total differential)

$$df(x,y) := \frac{\partial}{\partial x} f(x,y)dx + \frac{\partial}{\partial y} f(x,y)dy$$

is called the **total differential of** $f(x,y)$ **at the point** (x,y) and gives the total change of the height of the surface $f(x,y)$ (measured from the x–y–plane) at the point (x,y), if we move a tiny step dx along the x–direction and a tiny step dy along the y–direction.

Consider now a certain point (x_0,y_0) in the x–y–plane, with the gradient $\nabla f(x_0,y_0)$ of the function $f(x,y)$ attached. In that point, the total change of height of the function $f(x,y)$ can be written as a scalar product,

$$df = \frac{\partial}{\partial x} f dx + \frac{\partial}{\partial y} f dy = (f_x, f_y) \begin{pmatrix} dx \\ dy \end{pmatrix} = \nabla f \cdot \begin{pmatrix} dx \\ dy \end{pmatrix}$$

of the two vectors $(f_x, f_y) = \nabla f (\equiv \nabla f(x_0,y_0))$ and (dx,dy). We now change dx and dy slightly, thereby changing the vector (dx,dy) of the differentials. Then, for a certain values of dx and dy, the vector (dx,dy) becomes perpendicular to the gradient (f_x,f_y), i.e. the scalar product $df = \nabla f \cdot (dx,dy)$ vanishes. In this direction (dx,dy), the height of the surface does not change, it determines the direction of an equipotential line. Therefore, the gradient $\nabla f(x_0,y_0)$ is perpendicular to the equipotential line through (x_0,y_0); it determines the direction of the steepest increase of the function $f(x,y)$.

Example: Paraboloid $f(x,y) = x^2 + y^2$

We have

$$\nabla f(x,y) = (2x, 2y).$$

The equipotential lines are circles $r^2 = x^2 + y^2$ in the x–y–plane. The gradient is perpendicular to these circles. Picture in the lecture.