5. TWO–BY–TWO MATRICES

5.1 Two–by–Two Matrices: Introduction

5.1.1 Linear Equations of Two Unknowns

Consider the system of linear equations for the two unknowns x and y,

$$\begin{aligned} ax + by &= e \\ cx + dy &= f, \end{aligned} \tag{5.1}$$

where a, b, c, d, e, f are constant numbers. This system can be easily solved: solve the first equation for y,

$$y = \frac{e - ax}{b} \tag{5.2}$$

and insert it into the second equation,

$$\begin{aligned} cx + dy &= cx + \frac{de - adx}{b} = f \rightsquigarrow (cb - ad)x = fb - de \\ x &= \frac{de - fb}{ad - cb} \\ y &= \frac{e - ax}{b} = \frac{e(ad - cb) - a(de - fb)}{b(ad - cb)} = \frac{af - ec}{ad - cb}. \end{aligned} \tag{5.3}$$

For this general solution for x and y to be valid, the denominator $ad - cb$ apparently has to be different from zero.

5.1.2 Two–by–Two Matrices: Definition

We write the two unknowns x and y as the components of a two–dimensional vector $\mathbf{x}$,

$$\mathbf{x} := \begin{pmatrix} x \\ y \end{pmatrix}. \tag{5.4}$$

Then, we write the two constants e and f as the components of a two–dimensional vector $\mathbf{v}$

$$\mathbf{v} := \begin{pmatrix} e \\ f \end{pmatrix}. \tag{5.5}$$

The two–by–two system of linear equations, Eq. (5.1), **maps the vector x onto the vector v**. We write this in the following abstract form:

$$A\mathbf{x} \quad = \quad \mathbf{v} \Leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}, \tag{5.6}$$

where we defined the **two–by–two matrix**

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5.7}$$

A two–by–two matrix is a quadratic scheme which, upon operating on a vector $\mathbf{x}$ on its right, transforms this vector into another vector $\mathbf{v}$ according to the rule

$$A\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \mathbf{v}. \tag{5.8}$$

By comparison we recognise that this **matrix equation**, $A\mathbf{x} = \mathbf{v}$, is equivalent to the system Eq.(5.1).

5.1.3 Linear Mappings and Matrix Operatings

Definition: A *linear mapping* A from $R^2 \to R^2$ maps a vector $\mathbf{x}$ onto the vector $A\mathbf{x}$. The mapping is represented by a two-by-two matrix A. The mapping fulfills

$$\begin{aligned} \mathbf{x} &\to A\mathbf{x} && (5.9) \\ \mathbf{x}_1 + \mathbf{x}_2 &\to A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 && (5.10) \\ \lambda\mathbf{x} &\to A(\lambda\mathbf{x}) = \lambda A\mathbf{x}, \quad \lambda \in C. && (5.11) \end{aligned}$$

Examples

$$\begin{aligned} A &= \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_1 + \mathbf{x}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ \rightsquigarrow A\mathbf{x}_1 &= \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 3 \cdot (-2) \\ 2 \cdot 1 + (-1) \cdot (-2) \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \end{pmatrix} \\ A\mathbf{x}_2 &= \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}. && (5.12) \end{aligned}$$

We compare this to

$$\begin{aligned} A &= \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, \quad \mathbf{x}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}_1 + \mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ \rightsquigarrow A(\mathbf{x}_1 + \mathbf{x}_2) &= \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} = A\mathbf{x}_1 + A\mathbf{x}_2 \rightsquigarrow \text{OK}. \end{aligned}$$

5.2 Two–by–Two Matrices: Linear Mappings

Definition: The **determinant** $det(A)$ of a two–by–two matrix A is defined as

$$det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - cb. \tag{5.13}$$

5.2.1 Specific Linear Mappings 1: the Unit Matrix

This is the *trivial mapping* represented by the **unit matrix** E,

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.14}$$

We have $\det(E) = 1$. Check that $E\mathbf{x} = \mathbf{x}$ for any vector $\mathbf{x}$.

5.2.2 Specific Linear Mappings 2: Stretching and Shrinking

These are linear mappings A represented by the multiples of the **unit matrix**, where c is a real number such that

$$A = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}. \tag{5.15}$$

We have $\det(A) = c^2 > 1$. Check that in this case $A\mathbf{x} = c\mathbf{x}$ for any vector $\mathbf{x}$.

5.2.3 Specific Linear Mappings 3: Projections

These are linear mappings A such as

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.16}$$

We have $\det(A) = 0$. Check that in this case, for any vector $\mathbf{x} = (x, y)$, $A\mathbf{x} = (x, 0)$: the vector is projected onto the x-axis.

5.2.4 Specific Linear Mappings 4: Rotations

These are mappings $R(\theta)$ that rotate vectors around the origin by an angle θ,

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{5.17}$$

In this case, $\det(R(\theta)) = \cos^2\theta - (-\sin^2\theta) = 1$. A vector $\mathbf{x} = (x, y)$ is rotated into

$$R(\theta)\mathbf{x} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}. \tag{5.18}$$

Examples for rotations are

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \tag{5.19}$$

Special Rotations: $\theta = 0$

In this case,

$$R(\theta = 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E \quad \text{(unit matrix)}. \tag{5.20}$$

Special Rotations: $\theta = \frac{\pi}{2}$

In this case,

$$R\left(\theta = \frac{\pi}{2}\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_y \quad (-i \text{ times Pauli Matrix } \sigma_y). \tag{5.21}$$

5.2.5 Specific Linear Mappings 5: Reflections

These are mappings $S(\theta)$ that reflect a vectors at a fixed axis:

$$S(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{5.22}$$

In this case, $\det(S(\theta)) = -\cos^2\theta - \sin^2\theta = -1$. A vector $\mathbf{x} = (x, y)$ is transformed into

$$S(\theta)\mathbf{x} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ x\sin\theta - y\cos\theta \end{pmatrix}. \tag{5.23}$$

Examples:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix} = \begin{pmatrix} \cos\frac{1}{2}\theta\cos\theta + \sin\frac{1}{2}\theta\sin\theta \\ \cos\frac{1}{2}\theta\sin\theta - \sin\frac{1}{2}\theta\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix} \tag{5.24}$$

where we have a formula for trigonometric functions (CHECK). Furthermore, we have

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} \tag{5.25}$$

Sketch this in the x-y-plane (lecture). We recognise that $S(\theta)$ defines a reflection at the axis defined by the direction of the vector $(\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta)$

Special Reflection: $\theta = 0$

In this case,

$$S(\theta = 0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z \quad \text{(Pauli Matrix } \sigma_z\text{)}. \tag{5.26}$$

Special Reflection: $\theta = \frac{\pi}{2}$

In this case,

$$S\left(\theta = \frac{\pi}{2}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x \quad \text{(Pauli Matrix } \sigma_x\text{)}. \tag{5.27}$$

5.3 Two–by–Two Matrices: Index Notation and Multiplication

5.3.1 Basis Vectors and Index Notation

Vectors

Definition: The vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.28}$$

are called **basis vectors** of R^2. Any arbitrary vector $\mathbf{a} \in R^2$ is written as a **linear combination**

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 = \sum_{i=1}^{2} a_i\mathbf{e}_i. \tag{5.29}$$

In this representation, sometimes **Einstein's summation convention** is used: We write $\mathbf{a} = \sum_{i=1}^{2} a_i \mathbf{e}_i = a_i \mathbf{e}_i$, omitting the sum symbol in order to simplify the notation. The sum is automatically carried out over equal indices. Here, the index is i.

Matrices

Definition: The element A_{ij} of a matrix A is the entry in its i-th row and its j-th column. For two-by-two matrices, this reads

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{5.30}$$

Note: be very careful not to mix up the row and the column index!

Matrix operating on vector

The result of a linear mapping $\mathbf{x} \to \mathbf{y} = A\mathbf{x}$ can be written in index form, too:

$$\begin{aligned} \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &\to A\mathbf{x} = \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ \longleftrightarrow \quad y_i &= \sum_{j=1}^{2} A_{ij} x_j. \end{aligned} \tag{5.31}$$

This means that the first and second components, y_1 and y_2, of $\mathbf{y} = A\mathbf{x}$ are given by

$$y_1 = \sum_{j=1}^{2} A_{1j} x_j, \quad y_2 = \sum_{j=1}^{2} A_{2j} x_j. \tag{5.32}$$

Note that the index j runs over the columns of the matrix A.

5.3.2 Multiplication of a Matrix with a Scalar

This is simple,

$$\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}. \tag{5.33}$$

5.3.3 Matrix Multiplication: Definition

A matrix A moves a vector $\mathbf{x}$ into a new vector $\mathbf{y} = A\mathbf{x}$. This new vector can again be transformed into another vector $\mathbf{y}'$ by acting with another matrix B on it: $\mathbf{y}' = B\mathbf{y} = BA\mathbf{x}$. The combined operation $C = BA$ transforms the original vector $\mathbf{x}$ into $\mathbf{y}'$ in one single step. This **matrix product** is calculated according to

$$\begin{aligned} B &= \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \quad A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \\ \rightsquigarrow BA &= \begin{pmatrix} a_2a_1 + b_2c_1 & a_2b_1 + b_2d_1 \\ c_2a_1 + d_2c_1 & c_2b_1 + d_2d_1 \end{pmatrix}. \end{aligned} \tag{5.34}$$

In general, the matrix product does not commmute, i.e.,

$$AB \neq BA. \tag{5.35}$$

This means that in contrast to real or complex numbers, the result of a multiplication of two matrices A and B depends on the order of A and B.

Definition: The **commutator** $[A, B]$ of two matrices A and B is defined as

$$[A, B] = AB - BA. \tag{5.36}$$

The commutator plays a central role in quantum mechanics, where classical variables like position x and momentum p are replaced by **operators**(matrices) which in general do not commute, i.e., their commutator is non–zero.
Example:

$$\begin{aligned} \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_z\sigma_x &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_x\sigma_z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \neq \sigma_z\sigma_x, \quad [\sigma_z, \sigma_x] = 2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{5.37}$$

5.3.4 Matrix Multiplication: Index Notation

The abstract way to write a matrix multiplication with indices:

$$C = BA \rightsquigarrow C_{ij} = \sum_{k=1}^{2} B_{ik}A_{kj}. \quad (= B_{ik}A_{kj} \text{ in the summation convention}). \tag{5.38}$$

To get the element in the ith row and jth column of the product BA, take the scalar product of the ith row-vector of B with the j-th column vector of A. This looks complicated but it is not, it is just another formulation of our definition Eq.(5.34).

5.4 Inverse of a Matrix

5.4.1 Motivation

Solving the linear two–by–two system, Eq. (5.1, for the components x, y of the vector $\mathbf{x}$, is equivalent to the matrix equation

$$A\mathbf{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \mathbf{v} = \begin{pmatrix} e \\ f \end{pmatrix}. \tag{5.39}$$

We recognise that in order to explicitely solving this for $\mathbf{x}$, we have to **invert** the operation A.

5.4.2 Definition and Theorem

Definition: The **inverse** A^{-1} **of a two–by–two matrix** A is defined as the matrix fulfilling

$$A^{-1}A = AA^{-1} = \mathbf{1}, \quad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.40}$$

with the **unit matrix 1**.
Definition: The **determinant** $det(A)$ of a two–by–two matrix A is defined as

$$det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - cb. \tag{5.41}$$

Theorem Consider the two–by–two matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5.42}$$

If the determinant of A is non–zero, i.e. $det(A) = ad - cb \neq 0$, the inverse of A exists and is given by

$$A^{-1} = \frac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \equiv \begin{pmatrix} \frac{d}{ad-cb} & \frac{-b}{ad-cb} \\ \frac{-c}{ad-cb} & \frac{a}{ad-cb} \end{pmatrix}. \tag{5.43}$$

For the proof of this, we just multiply A with A^{-1} and A^{-1} with A:

$$AA^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - cb} \begin{pmatrix} ad - bc & -ab + ba \\ cd - dc & -cb + da \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.44}$$

Exercise: Check the same for $A^{-1}A$.

Examples

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \rightsquigarrow det(A) = -1 - 6 \neq 0, \quad A^{-1} = \frac{1}{-7}\begin{pmatrix} -1 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{2}{7} & \frac{-1}{7} \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix} \rightsquigarrow det(A) = 3 \cdot 4 - 2 \cdot 6 = 0 \rightsquigarrow A^{-1}\text{does not exist.}$$

Solving the Linear Equations (5.1)

We are now in a position to solve Eq. (5.1) by the inverse of a matrix:

$$\begin{aligned} A\mathbf{x} &= \mathbf{v} \Leftrightarrow A^{-1}A\mathbf{x} = A^{-1}\mathbf{v} \Leftrightarrow \mathbf{x} = A^{-1}\mathbf{v} \\ \rightsquigarrow \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{ad - cb}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} \frac{de-bf}{ad-bc} \\ \frac{-ce+af}{ad-bc} \end{pmatrix}. \end{aligned} \tag{5.45}$$