1.1 Curves (don't use a calculator)

1.1.1 Lorentzian

a) Sketch the curves

$$f_a(x) = \frac{1}{x^2 + a^2}, \quad f_b(x) = \frac{1}{x^2 + b^2}, \quad f_c(x) = \frac{1}{x^2 + c^2} \tag{1.1}$$

for $a > b > c > 0$ within one coordinate system.
b) Calculate the constant N in $f(x) = N/[x^2 + a^2]$ such that $\int_{-\infty}^{\infty} dx f(x) = 1$.

1.1.2 Gaußian

Sketch the curves

$$g_a(x) = e^{-x^2/a^2}, \quad g_b(x) = e^{-x^2/b^2}, \quad g_c(x) = e^{-x^2/c^2} \tag{1.2}$$

for $a > b > c > 0$ within one coordinate system.

1.1.3 Crazy Fractions

Sketch the curves (one graph each)

$$s(x) = \sin\left(\frac{1}{x}\right), \quad t(x) = \ln\left(\frac{1}{x}\right), \quad u(x) = \exp\left(-\frac{1}{x^2}\right). \tag{1.3}$$

1.1.4 Modulus

Sketch the functions $a_i(t), i = 1, 2, 3$ (all into one graph)

$$a_1(t) = |t|, \quad a_2(t) = |t - 1|, \quad a_3(t) = |t + 1|. \tag{1.4}$$

1.1.5 Black Body Radiation

1) Sketch the Planck distribution for the spectral energy density of black body radiation as a function of the frequency ν,

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}. \tag{1.5}$$

Sketch different curves (each with one fixed temperature T).
2) Show that the total energy density $U(T) = \int_0^{\infty} d\nu u(\nu, T)$ is proportional to T^4 (Stefan–Boltzmann–law).

1.2 Parametric curves

A particle moves in the x–y–plane on a curve with coordinate $x(t)$ and $y(t)$, where the parameter t denotes the time. t shall run from $t = 0$ up to $t = \infty$. Sketch the following curves, including direction arrows:

$$x(t) = 1, \quad y(t) = t \tag{1.6}$$
$$x(t) = t, \quad y(t) = 2t \tag{1.7}$$
$$x(t) = a\sin(\omega t), \quad y(t) = b\cos(\omega t), \quad \omega > 0, a > 0, b > 0 \tag{1.8}$$
$$x(t) = \exp(-\gamma t)\sin(\omega t), \quad y(t) = \exp(-\gamma t)\cos(\omega t), \quad \omega > 0, \gamma > 0 \tag{1.9}$$
$$x(t) = \exp(-\gamma t)\sin(\omega t), \quad y(t) = \exp(-\gamma t)\cos(\omega t), \quad \omega > 0, \gamma < 0. \tag{1.10}$$

1.3 Exponential Equations

We define the function $\sinh(x) := (1/2)\left(e^x - e^{-x}\right)$ (hyperbolic sine). Inverting $y = \sinh(x) \to x = \sinh^{-1}(y)$, we find the inverse hyperbolic sine (area sinus hyperbolicus) $\sinh^{-1}$ by setting $y = (e^x - e^{-x})/2 \rightsquigarrow e^{2x} - 2ye^x - 1 = 0$. Solve this equation for x and show that $\sinh^{-1}(y) = x = \ln\left(y + \sqrt{y^2+1}\right)$.

1.4 Integrals

Calculate
a) $\int dx \ln(x)$; b) $\int dx x e^{ax}$; c) The volume of the solid of revolution if the curve $y(x) = 1 - x^2$, $-1 < x < 1$ is rotated around the y–axis; d) $f'(x)$ where $f(x) = \int_0^x dy g^2(y)$ for a general $g(y)$.

1.5 Differential Equations; time–dependent rates and sources

a) Solve $y'(x) = e^{2x}, y(0) = 1$. b) Solve $y'(x) = 1/\ln(y), y(0) = 1$.
c) Consider an amount of mass $y(t)$ that varies as a function of time according to $y'(t)+r(t)y(t) = s(t)$, where $r(t)$ is the time–dependent rate of increase (or decrease) of the mass and $s(t)$ is a time–dependent mass source. Show that the mass at time t for the initial condition $y(0) = 0$ is given by

$$y(t) = \frac{1}{g(t)}\int_0^t dt' s(t')g(t'), \quad g(t) := e^{\int_0^t dy r(y)}. \tag{1.11}$$

Hint: write $1/g(t) = e^{-\cdots}$ and verify the expression for $y(t)$ by calculating $y'(t)$. Discuss limiting cases of this general formula, such as $s(t) = 0$, $r(t) = r = const.$
d) Solve $y'(x) - \tan(x)y(x) = \cos(x), y(0) = 0$. Hint: you can use part c).