10.1 Physics

10.1.1 Motion of a charge

A charge $-e$ is moving in the x-y-plane in the potential $\Phi(x,y) = \frac{c}{\sqrt{x^2+y^2}}$, where c is a constant. Make a sketch of $\Phi(x,y)$!
1. Calculate the change $d\Phi(x,y)$ (total differential) of the potential if the charge moves from (x,y) to $(x+dx, y+dy)$.
2. Calculate the change in energy, $-e\frac{d}{dt}\Phi(x(t),y(t))$, if the charge moves a) along a circle $(r\cos\omega t, r\sin\omega t)$, b) along a straight, radial line $(a+vt, a+vt), a, v, t > 0$.

10.2 Math Practise

10.2.1 Partial Derivatives

Calculate 1. $\frac{\partial}{\partial x}\cosh(x^2+y^2)$; 2. $\frac{\partial}{\partial y}\sinh[\sin(xy)]$.

10.2.2 Gradient of $f(x,y) = x^2 + y^2$.

a) Calculate the gradient $\nabla f(x,y)$ of $f(x,y) = x^2 + y^2$.
b) Sketch the 'vector field' of the gradients in the x-y-plane: make a sketch by 'attaching' small arrows $\nabla f(x_0, y_0)$ at points (x_0, y_0).
c) From the sketch in b), sketch the equipotential lines of $f(x,y)$.

10.2.3 Gradient of $f(x,y) = x^2 - y^2$.

a) Calculate the gradient $\nabla f(x,y)$ of $f(x,y) = x^2 - y^2$.
b) Sketch the 'vector field' of the gradients in the x-y-plane: make a sketch by 'attaching' small arrows $\nabla f(x_0, y_0)$ at points (x_0, y_0).
c) From the sketch in b), sketch the equipotential lines of $f(x,y)$.

10.3 Math Problems

10.3.1 Stationary Points of Functions $f(x,y)$

A) Find the stationary points (x,y) (if there are any), defined by

$$\nabla f(x,y) = (0,0) \rightarrow (x,y) \quad \text{is a stationary point}$$

of the following functions:
1. $f(x,y) = y$; 2. $f(x,y) = x$; 3. $f(x,y) = -(x^4 + y^4)$; 4. $f(x,y) = xy$; 5. $f(x,y) = x^4 - y^4$.

*B) Discuss the case 4, $f(x,y) = xy$, in some more detail by sketching the 'vector field' $\nabla f(x,y)$ in the x-y-plane.

10.3.2 Taylor-expansion in two dimensions

The lowest order Taylor-expansion (up to the linear terms) of a function $f(x,y)$ around $(x = x_0, y = y_0)$ is, in analogy with a function of one variable, given by

$$f_1(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y - y_0).$$

Show that $f_1(x,y)$ defines a two-dimensional *plane* and sketch (or demonstrate in a 'three-dimensional model') this plane for $f(x,y) = x^2 + y^2$, $(x_0,y_0) = (0,1)$.

10.3.3 *Differential Equation, Graphical Representation

Consider the differential equation $y'(x) = -\frac{x}{y(x)}$.
a) Make a sketch in the upper x-y-halfplane: to each point (x_0,y_0), attach a small piece of a straight line (arrow) with slope $-\frac{x_0}{y_0}$.
b) Argue how this sketch can be used to graphically solve the differential equation.
c) Solve the differential equation exactly, and show that b) is consistent with that solution.

10.3.4 * Second order surfaces

Discuss and sketch the following:

$$\begin{aligned}
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= 1 \quad \text{ellipsoid}, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \text{hyperboloid type 1}, \\
\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} &= -1 \quad \text{hyperboloid type 2}, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \quad \text{cone}, \\
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= z \quad \text{elliptical paraboloid}, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = z \quad \text{hyperbolic paraboloid}.
\end{aligned}$$

10.3.5 Revision: Differential Equations

Consider the differential equation $y''(x) + 4y(x) = 2e^{i3x}$.
a) Write down the general solution of the homogeneous equation, using exponentials.
b) Write a particular solution $y_p(x)$ of the inhomogeneous equation as $y_p(x) = Ce^{i3x}$ and determine the constant C.
c*) Determine the general solution of the differential equation, and solve it for $y(0) = 1$, $y'(0) = 0$.