12.1 Math Practice

Consider the following matrices:

$$A = \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & c \\ c & 0 \end{pmatrix}, \quad C = \begin{pmatrix} z & 1 \\ 1 & z^* \end{pmatrix}, \quad P_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

12.1.1 Matrix Inverse

For which of the above matrices does the inverse exist? Calculate the inverse if it exists.

12.1.2 Matrix Multiplication, Commutator

Calculate products and commutators (recall that the commutator of two matrices X and Y is $[X,Y] = XY - YX$), for the following:
a) AB, BA, $[A,B]$; b) $\sigma_x\sigma_y$, $\sigma_y\sigma_x$, $[\sigma_x, \sigma_y]$.

12.1.3 Special Case: Rotations in Two Dimensions

Show that rotations $R(\theta)$ in two dimensions commute, i.e., $R(\theta_1)R(\theta_2) = R(\theta_2)R(\theta_1)$.
a) Prove this by a geometric argument (sketch!)
b) Prove this by calculating $R(\theta_1)R(\theta_2)$ for arbitrary θ_1 and θ_2 and using the theorems

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right], \quad \cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right].$$

12.2 Math Problems

12.2.1 Exponential Function of a Matrix

The exponential function of a matrix A is defined as $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$. Consider the matrix

$$A = \lambda\sigma_z = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix},$$

use the definition $A^0 = I$ (unit matrix), calculate A^n for $n = 1, 2, 3, 4, \ldots$, and from this calculate e^A.

12.3 Revision: MATH II

12.3.1 Quadratic Equations: solve $z^2 - 10z + 40 = 0$.

12.3.2 Calculate and sketch $f_r(x) := \mathrm{Re}[f(x)]$ and $f_i(x) := \mathrm{Im}[f(x)]$ for $f(x) = [x+i]^{-1}$ (real x).

12.3.3 Sketch the region of complex numbers in the complex plane (x–y plane) with $1 < |z| < 2$.

12.3.4 Calculate $f(t) \equiv \mathrm{Re}(e^{i\Omega t})$ where $\Omega = \omega + i\gamma$ and t, ω and γ are real.

12.3.5 Write down the infinite series which define Sine, Cosine, and the Exponential Function.

12.3.6 Calculate the derivative $\tanh'(x)$.

12.3.7 Solve $y'(x) = e^{px}, y(0) = 1, p \in R$.

12.3.8 Find the general solution of $y''(x) - y'(x) - 2y(x) = 0$. Find the solution that fulfills $y(0) = 0$ and $y'(0) = 1$.

12.3.9 Write down the general solution of $y''(x) + ay(x) = 0, a > 0$, written in terms of real functions.

12.3.10 Find the general solution of a) $y''(x) - 4y'(x) + 5y(x) = 0$; b) $y''(x) + y'(x) + 12y(x) = 0$; c) $y''(x) - ay(x) = 0, a > 0$.

12.3.11 Sketch $f(x) = x\sin(x)/(1+x^2)$.

12.3.12 Calculate $1 + 2a + 4a^2 + 8a^3 + 16a^4 + ...$ Which condition must be fulfilled for this expression to converge?

12.3.13 Taylor expand (3 terms) around $x = 0$: a) $f(x) = \sqrt{2+x}$; b) $f(x) = \sin(x)/x$.

12.3.14 Taylor expand (all terms) around $x = 0$: a) $f(x) = e^{4+x^4}$; b) $f(x) = 1/(1+x^6)$.

12.3.15 Approximately calculate $\sqrt{100-a}$, a arbitrary with $|a| < 10$.

12.3.16 Calculate $\lim_{x\to 0} x^2/[\sin(ax^2)], a > 0$.

12.3.17 Calculate $\frac{\partial^2}{\partial y \partial x}\sin(x + x^2y^3)$.

12.3.18 Sketch the equipotential lines ($f = const$) and calculate the gradient and stationary points of a) $f(x,y) = x^4 + y^4$; b) $f(x,y) = x^4 - y^4$.

12.3.19 Calculate (see page 1) a) the determinant of B and C; b) BC; c) C^{-1}.