2.1 Physics

1. A quantum particle moving in a thin wire has a wave function

$$\Psi(x) = \frac{c}{x+i},$$

where x is the coordinate of the particle and c a positive constant. Calculate and sketch the probability function

$$p(x) = \Psi^*(x)\Psi(x)$$

of the particle ($p(x)dx$ describes the probability to find the particle in the small interval dx around x). Where is the maximum of this probability?

2. Consider an AC circuit with external complex voltage $V(t) = V_0 e^{i\omega t}$ and complex current $I(t) = I_0 e^{i\omega t}$. The elements of a general circuit are resistors R, inductors L, and capacitors C. The admittance Y_R of a resistor R is defined as $Y_R = 1/R$, the admittance Y_L of an inductor L is defined as $Y_L = -i/(\omega L)$, and the admittance Y_C of a capacitor C is defined as $Y_C = i\omega C$.

a) In a circuit with R and L parallel (see figure), the total admittance Y is the sum of the two admittances, and the complex current amplitude I_0 is $I_0 = YV_0$ with V_0 being the complex voltage amplitude. Calculate the modulus $|I_0|$ of the current amplitude for this circuit.

b) The complex resistance of a capacitor C is $Z_C = 1/Y_C = 1/i\omega C$. The total complex resistance Z of a circuit with a resistor R and a capacitor C in series is $Z = R + Z_C$. Sketch the circuit and calculate the total complex current amplitude I_0 from $V_0 = ZI_0$, where V_0 is the total complex voltage amplitude. Then calculate the complex voltage drop V_C at the capacitor and its modulus $|V_C|$. Identify a characteristic time-scale of this circuit.

2.2 Math Practise

2.2.1 Basic Operations

Calculate the following:

a) $(-1+2i)(4-i)$

b) $(2-i)(-7+22i)$

c) $\frac{3-2i}{-1+i}$

d) i^4
e) i^{30}
*f) Find real numbers x and y such that $3x + 2iy - ix + 5y = 7 + 5i$.

2.2.2 Quadratic Equations, Real and Imaginary Part

a) Find the complex solutions of $z^2 - 10z + 40 = 0$.

b) Define the function $f(x) = [x+i]^{-1}$ for real x. Calculate and sketch $f_r(x) := Re[f(x)]$ and $f_i(x) := Im[f(x)]$.

c) Show that for any complex number Ψ, the product $\Psi^*\Psi$ (where Ψ^* is the conjugate complex of Ψ) is real.

2.3 Math Problems

2.3.1 Vector and Polar Form

Let $z_1 = 2 + i$, $z_2 = 3 - 2i$.
a) Draw z_1 and z_2 as vectors in the complex plane. Calculate $z = z_1 + z_2$ and draw z as a vector in the complex plane.
b) Calculate $|z_1|$ and $|z_2|$ and explain their meaning.
c) Prove graphically that $|z_1 + z_2| \leq |z_1| + |z_2|$. Confirm this by direct calculation.
d) What is the polar form for the complex number $z = x + iy$, x and y real?
e) Express $z = 1 + i$ in polar form. Check the result by drawing z as a vector in the complex plane.
f) Sketch the region of complex numbers in the complex plane (x–y plane) with $1 < |z| < 2$.

2.3.2 De Moivre's Theorem

a) Take the square of a complex number z in polar form and prove two identities for trigonometric functions from that.