3.1 Physics

3.1.1 Damped Oscillator

Consider the weakly damped, harmonic oscillator as described by

$$\ddot{x}(t) + \gamma \dot{x}(t) + \omega^2 x(t) = 0, \quad \omega \gg \gamma > 0.$$

Assume $x(t) = e^{izt}$ and insert this into the differential equation. Solve the resulting quadratic equation in order to obtain two values $z = z_1$ and $z = z_2$. Discuss the corresponding solutions $e^{iz_1 t}$ and $e^{iz_2 t}$.

3.1.2 The quantum Hall effect

The quantum Hall effect occurs in a two–dimensional sheet (x–y–plane) of electrons in a strong magnetic field B perpendicular to the plane. Non–interacting electrons are described by complex coordinates $z = x + iy$ in the x–y–plane, with wave functions

$$\Psi_m(z) = \frac{1}{\sqrt{2^{m+1}\pi m!}} \left(\frac{z}{l_B}\right)^m e^{-\left|\frac{z}{2l_B}\right|^2},$$

where $l_B = (c/|eB|)^{1/2}$ is the typical lengthscale of this systems which is called 'magnetic length'. Sketch the probability $|\Psi_m(z)|^2$ as a function of $|z|$ for different quantum numbers m. Large m corresponds to large angular momentum: argue why this is consistent with your plots.

3.2 Math Practise

3.2.1 General

a) Express the real and imaginary part of a complex number z, using z and z^*.
b) Express sine and cosine in terms of the complex exponential function.
c) Sketch the following functions (x real): $\ln(x)$, $\exp(-x)$, $\exp(-x^2)$, $1/(1+x^4)$.
d) Prove that $(1/z)^* = 1/z^*$ for any complex number z.

3.2.2 Polar Representation

Calculate the following, expressing everything in the polar representation
a) $\sqrt{i}$; b) $\sqrt{1+i}$; c) $(1+i)^2$; d) $(1+i)^4$.

3.2.3 The Complex Plane (Argand Diagram)

a) Why do inequalities like $z_1 < z_2$ make no sense for complex numbers?
b) Sketch the area of the complex plane with numbers z fulfilling $|z - 2i| < 1$.

3.2.4 The Complex Exponential

a) Write down the definition of the exponential function in terms of an infinite series.
b) Express $\mathrm{Re}(e^{-ix})$ and $\mathrm{Im}(e^{-ix})$ by real functions.
c) Calculate $e^{(\pi/2)i}$, $e^{\pi i}$, $e^{2\pi i}$.
d) Calculate $f(t) \equiv \mathrm{Re}(e^{i\Omega t})$ where $\Omega = \omega + i\gamma$ and t, ω and γ are real. What is the limiting value of $f(t)$ for $t \to \infty$ and positive γ? What is the limiting value of $f(t)$ for $t \to \infty$ and negative γ?

3.3 Math Problems

3.3.1 Complex Numbers

a) Calculate $e^{(\pi/4)i}$
b) Calculate i^i.

3.3.2 Sine and Cosine

Consider the infinite series for $\sin(x)$ (also cf. lecture notes)

$$\sin(x) = x - x^3/3! + x^5/5! - x^7/7! \pm ...$$

a) Find a corresponding series for $\cos(x)$ by differentiation.
* b) Using the series, verify that $y(x) = \sin(x)$ fulfills the second order ordinary differential equation

$$y''(x) + y(x) = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

c) Using the series for Sine and Cosine, find an approximation for both $\sin(x)$ and $\cos(x)$ for very small values of the argument $x \ll 1$. Sketch this approximation graphically.

3.3.3 * Laplace Transformation

In order to analyse a function $f(t)$, we introduce its Laplace transformation

$$\tilde{f}(z) \equiv \int_0^\infty e^{-zt} f(t) dt,$$

where z is a complex variable. Calculate the Laplace transformation of the function

$$f(t) = e^{-\gamma t} \cos(\omega t), \quad \gamma > 0$$

which describes a damped oscillation. Hint: Use the Euler formula to decompose the Cosine. Find the poles (singularities) of $\tilde{f}(z)$ and discuss the 'physical content' of these poles.