4.1 Math Practise (no calculators allowed)

4.1.1 **Complex Numbers**: Calculate the inverse $1/z$ and the polar form $z = re^{i\phi}$ of the following:

a) $z = i$; b) $z = -i$; c) $z = -1 + i$.

4.1.2 **Complex Numbers**: Calculate the following in Cartesian form $z = x + iy$.

a) $z = e^{\pi i}$; b) $z = 2e^{\pi i}$; c) $z = e^{(2n+1/2)\pi i}, n = 0, 1, 2, ...$

4.1.3 Hyperbolics

a) Write down the definition of $\sinh(x)$, $\cosh(x)$, $\tanh(x)$, and $\coth(x)$.
b) Calculate the values of the functions in a) for $x = 0$. Sketch the functions in a).
c) Calculate the derivative $\tanh'(x)$.
*d) find an approximation for $\ln[\sinh(x)]$ for very large $x \gg 1$.

4.1.4 Hyperbolics and Trigonometric Functions

a) For real x, simplify $\sinh(ix)$, $\cosh(ix)$, and $\tan(ix)$.
b) Simplify $\cosh(z) - \sinh(z)$; c) Calculate $|1 - e^{ix}|^2$ for real x.

4.1.5 Inverse Hyperbolics

a) Sketch the inverse hyperbolic cosine, $\cosh^{-1}(x)$.
*b) In analogy to the inverse hyperbolic sine (lecture), express the positive branch of $\cosh^{-1}(x)$ in terms of a logarithm.

4.1.6 Differential Equations

Classify the following (linear or nonlinear, homogeneous or inhomogeneous):
a) $y'' + y = x^2$; b) $y'' + y^2 = x$; c) $y'' + |y| = 0$; d) $y'' + xy = 1$.

4.2 Physics

4.2.1 Damping

A particle of mass m moves on a line (x–axis). The only force acting on the particle is a friction $-\gamma\dot{x}(t)$ ($x(t)$: position of the particle at time t.)
a) Write down Newton's law for this problem.
b) Solve the resulting differential equation. Hint: solve for the velocity first. Assume that a time $t = 0$, the position is $x(t = 0) = x_0$ and the velocity is $v(t = 0) = v_0$.

4.2.2 Statistical Mechanics of a Spin

An electron spin 1/2 is a magnetic field $(0,0,B)$ parallel to the z–axis has two possible energies $E_1 = +\alpha B$ (spin up) and $E_2 = -\alpha B$ (spin down), where $\alpha > 0$ a constant.

The spin is in thermal equilibrium at temperature T, where the average $\bar{E}$ of its energy is expressed by a sum over exponential Boltzmann weights w_n as

$$\bar{E} = \frac{E_1 w_1 + E_2 w_2}{w_1 + w_2}, \quad w_n \equiv e^{\frac{-E_n}{k_B T}}, \quad n = 1, 2. \tag{4.1}$$

Here, k_B is the Boltzmann constant.
a) Show that the *probabilities* $p_n = w_n/(w_1 + w_2)$ fulfill $\sum_{n=1}^{2} p_n = 1$.
b) Calculate the 'partition sum' $Z = w_1 + w_2$ of all Boltzmann weights. Express the result in terms of a known function.
c) Calculate $\bar{E}$ explicitely (in terms of a known function).
d) Sketch $\bar{E}$ as a function of the magnetic field B for different temperatures T. Discuss your results: what happens at very large and at very low temperatures ?

4.3 Math Problems

4.3.1 Differential Equations: 2nd order → 1st order

a) Show that $y''(x)+p(x)y'(x) = f(x)$ can be transformed into a first order equation. Derive that equation and solve it for $f(x) \equiv 0$.
b) Solve $y'(x) = e^{2x}, y(0) = 1$.

4.3.2 Differential Equations, integrating factor

a) Consider a star of mass $y(t)$ that varies as a function of time according to $y'(t) + r(t)y(t) = s(t)$, where $r(t)$ is the time–dependent rate of increase (or decrease) of the mass and $s(t)$ is a time–dependent mass source. Show that the solution at time t for the initial condition $y(0) = 0$ can be written using the integrating factor $g(t)$,

$$y(t) = \frac{1}{g(t)} \int_0^t dt' s(t') g(t'), \quad g(t) = e^{\int_0^t dy r(y)}. \tag{4.2}$$

Hint: write $1/g(t) = e^{-\cdots}$ and verify the expression for $y(t)$ by calculating $y'(t)$. Discuss limiting cases of this general formula, such as $s(t) = 0$, $r(t) = r = const.$
b) Solve $y'(x) - \tan(x)y(x) = \cos(x), y(0) = 0$. Hint: you can use part a).