5.1 Physics

5.1.1 Motion of a Ball

A ball of mass m moves on a line (x–axis) is attached to two springs (figure). The resulting force acting on the ball is $-kx(t), k > 0$.
a) Write down the equation of motion (Newton's law) for the ball. Write this equation in the standard form $\ddot{x}(t) + ... = ...$
b) Write down two independent solutions $x_1(t) = ...$ and $x_2(t) = ...$
c) Show that these solutions contain the dimensionless variable ωt with $\omega = \sqrt{k/m}$. What is the physical meaning of ω?

5.1.2 Friction

A ball of mass m rolls on a track (x–axis), with the only force acting on it being a friction force $-\gamma v(t)$ ($v(t)$: velocity of the ball at time t.)
a) Write down Newton's law as a differential equation for the velocity $v(t)$.
b) Solve this differential equation and determine the position $x(t)$ of the ball, assuming that a time $t = 0$, the position is $x(t = 0) = x_0$ and the velocity is $v(t = 0) = v_0$.

5.1.3 Potentials and Forces

A particle of mass m moves on a line (x–axis) under the influence of a force $F(x)$, where x is the position of the particle. The function $V(x)$ with $F(x) = -V'(x)$ is called the potential of the force.

Derive the equation of motion of the particle in the standard form $\ddot{x}(t) + ... = 0$ for the following potentials, and classify the equation as linear/non–linear:
a) $V(x) = cx, c > 0$; b) $V(x) = -(k/2)x^2, k > 0$; c) $V(x) = V_0 \sin(x)$
* d) For which integer n does the potential $V(x) = \alpha x^n$ lead to a linear equation of motion? Sketch $V(x)$ for positive and negative α for that n, and sketch typical curves $x = x(t)$.

5.1.4 Alice

Alice has a mass m and falls into an infinitely deep well (z–axis). The two forces acting on her are the gravity force $F_g = -mg, g > 0$, and the friction force $-\gamma w(t), \gamma > 0$, where $w(t)$ is the velocity of Alice at time t (all in z–direction).
a) Write down Newton's law for this problem.
* b) Solve the resulting differential equation. Assume that a time $t = 0$, the position of Alice is $z(t = 0) = 0$ and her velocity is $w(t = 0) = w_0$.

* c) Determine her 'stationary' velocity $w(t \to \infty)$ from the solution of the differential equation. Check that this is correct by balancing forces.

5.2 Math Practise + Problems

5.2.1 Differential Equations $y''(x) - y'(x) - 2y(x) = 0$

a) Show that this differential equation has two different solutions of the form $y(x) = e^{\alpha x}$, and determine the two possible values for α. Hint: insert $e^{\alpha x}$ into the differential equation.
b) Show that $y(x) = c_1 e^{-x} + c_2 e^{2x}$ is the general solution of $y''(x) - y'(x) - 2y(x) = 0$.
c) Determine c_1 and c_2 such that $y(0) = 0$ and $y'(0) = 1$.

5.2.2 Differential Equation $y''(x) + y(x) = 0$

a) Check that this differential equation has two different (linearly independent) solutions of the form $y_1(x) = e^{ix}$, $y_2(x) = e^{-ix}$.
b) Check that this differential equation has two (linearly independent) solutions of the form $v_1(x) = \sin(x)$, $v_2(x) = \cos(x)$.
c) Show that v_1 linearly depends on y_1 and y_2, and that v_2 linearly depends on y_1 and y_2 by expressing v_1 and v_2 as linear combinations of y_1 and y_2.
d) Use the linear combination $c_1 \sin(x) + c_2 \cos(x)$ to solve this differential equation with the initial condition $y(0) = 1$, $y'(0) = 0$.

5.2.3 Differential Equation $y''(x) - y(x) = 0$

a) Show that this differential equation has two different solutions of the form $y_1(x) = e^{x}$, $y_2(x) = e^{-x}$.
b) Show that this differential equation has two different solutions of the form $v_1(x) = \sinh(x)$, $v_2(x) = \cosh(x)$.
c) Show that one can express v_1 and v_2 as linear combinations of y_1 and y_2.
d) Use the linear combination $c_1 \sinh(x) + c_2 \cosh(x)$ to solve this differential equation with the initial condition $y(0) = 0$, $y'(0) = 1$.

5.2.4 Differential Equation $y''(x) - 4y'(x) + 5y(x) = 0$

a) Show that this differential equation has two different solutions of the form $y_1(x) = e^{\alpha_1 x}$ and $y_2(x) = e^{\alpha_2 x}$ and determine α_1 and α_2
b) Write the general solution of this differential as $y(x) = c_1 y_1(x) + c_2 y_2(x)$ and show that this can be written as $y(x) = e^{2x}[d_1 \cos(x) + d_2 \sin(x)]$ with new constants d_1 and d_2.