6.1 Physics

6.1.1 RLC–circuit

Consider a closed circuit with a resistor R, an inductor L, and a capacitor C in series (figure). The time–dependent charge $Q(t)$ on the capacitor, the corresponding voltage drop $U(t)$, and the current $I(t)$ through the circuit are related by

$$Q(t) = CU(t), \quad I(t) = -\dot{Q}(t). \tag{6.1}$$

Furthermore, the voltage $U(t)$ must be equal to the sum of the two voltage drops $L\dot{I}(t)$ (inductor) and $RI(t)$ (resistor), i.e.,

$$U(t) = L\dot{I}(t) + RI(t). \tag{6.2}$$

1. Use these two equations to eliminate the charge $Q(t)$ and the current $I(t)$ in order to find a differential equation for $U(t)$. Show that this differential equation reads

$$\ddot{U}(t) + \frac{R}{L}\dot{U} + \frac{1}{LC}U(t) = 0. \tag{6.3}$$

2. Classify 6.3 (nonlinear/linear ?, homogeneous/inhomogeneous ?)
3a. Find two independent solutions of 6.3 **for the special case** $R = 0$ (zero resistance) with our 'secret weapon' (insert $\exp(izt)$ and find possible values for z).
3b. Show that the general solution can be written as $U(t) = [A\cos(\omega t) + B\sin(\omega t)]$ with $\omega = \sqrt{1/LC}$.
4a. Find two independent solutions of 6.3 **for the general case** $R \neq 0$ (finite resistance) with our 'secret weapon' (insert $\exp(izt)$ and find possible values for z). You can assume that $R < 2\sqrt{L/C}$.
4b. Show that the general solution can be written as $U(t) = e^{\alpha t}[A\cos(\omega t) + B\sin(\omega t)]$ with

$$\alpha = \frac{R}{2L}, \quad \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}. \tag{6.4}$$

4c. Show that your solution in part 4. is consistent with your solution in part 3.

6.1.2 RLC–circuit with external voltage

Consider the RLC circuit with the external voltage $V(t) = V_0\sin(\Omega t), V_0 > 0$ (figure). The voltage $V(t)$ must fulfill

$$L\dot{I}(t) + RI(t) = \frac{1}{C}Q(t) + V(t), \tag{6.5}$$

where $I(t) = \dot{Q}(t)$ is the time–dependent current through the circuit.
1. Find the differential equation for $I(t) = -\dot{Q}(t)$ (hint: differentiate this equation),

$$L\ddot{I}(t) + R\dot{I}(t) + \frac{1}{C}I(t) = \Omega V_0 \cos(\Omega t). \tag{6.6}$$

2. Classify 6.6 (nonlinear/linear ?, homogeneous/inhomogeneous ?)
3. Consider the 'auxiliary' differential equation to (6.6), where $\cos(\Omega t) = \mathrm{Re}\, e^{i\Omega t}$ is replaced by $e^{i\Omega t}$,

$$L\ddot{I}^a(t) + R\dot{I}^a(t) + \frac{1}{C}I^a(t) = \Omega V_0 e^{i\Omega t}, \tag{6.7}$$

and solve this equation by inserting $I^a(t) = I_0 e^{i\Omega t}$ and determine I_0.
4a. Calculate the polar form of the complex number $I_0 = |I_0|e^{i\phi}$ (note that $V_0 > 0$ is real!),
4b*. Prove the following: i) If the complex current $I^a(t)$ is a solution of (6.7), the complex conjugate $[I^a(t)]^*$ is a solution of (6.7) with $e^{i\Omega t}$ replaced by $e^{-i\Omega t}$, and ii) consequently, the 'true' current $I(t) = \mathrm{Re}\, I^a(t)$ is a solution of (6.6).
4c. Use 4a to find the 'true' current $I(t) = \mathrm{Re}\, I^a(t) = \mathrm{Re}\, I_0 e^{i\Omega t} = |I_0|\cos(\Omega t + \phi)$, i.e. the solution of (6.6). What is the value of ϕ in the case $R = 0$ (zero resistor), and what is the phase shift between the current $I(t)$ and the voltage $U(t)$ in that case ?

6.2 Math Practise

6.2.1 Homogeneous Differential Equation

Find the general solution of a) $y''(x) + 12y(x) = 0$; b) $y''(x) + y'(x) + 12y(x) = 0$.

6.2.2 Sketch the following Curves (normal stream)

$$f(x) = \frac{1}{1+x^2} \text{ (LORENTZIAN)}, \quad f(x) = e^{-x^2} \text{ (GAUSSIAN)}, \quad f(x) = \frac{\sin(x)}{1+x^2}.$$

6.3 Math Problems

6.3.1 Inhomogeneous Differential Equation

a) Consider the differential equation $y''(x) + 4y(x) = 2e^{i3x}$. Write a particular solution $y_p(x)$ of this inhomogeneous equation as $y_p(x) = Ce^{i3x}$ and determine the constant C.
b*) Determine the general solution of this differential equation, and solve it for $y(0) = 1$, $y'(0) = 0$.