8.1 Math Practise

8.1.1 Truncated Taylor Expansions

Find the first three terms in the Taylor expansion for small $|x|$ around $x = 0$ of the following functions:
1. $f(x) = \sqrt{1+x}$; 2. $f(x) = \arctan(x)$; 3. $f(x) = \frac{1}{\sqrt{1+x}}$;
4. $f(x) = \frac{1}{\sqrt{1-x}}$; 5. $f(x) = \frac{\sin(x)}{x}$

8.1.2 Taylor Expansions

Find the Taylor expansion (all terms) for small $|x|$ around $x = 0$ of the following functions:
1.
$$f(x) = \arctan(x).$$
2.
$$f(x) = \frac{1}{1+x}.$$
3.
$$f(x) = \cosh(x).$$

8.1.3 Approximations

1. Use the Taylor expansion of $f(x) = \sqrt{1+x}$ to approximately calculate $\sqrt{10}$. Hint: Write $10 = 9 + 1 = 9(1 + 1/9)$.
2. Approximately calculate a) $\sqrt{143}$ and b) $\sqrt{100-a}$, $|a| < 10$.

8.2 Math Problems

8.2.1 Taylor Expansions

1. Use $\ln \frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$ and show that for $|x| < 1$,
$$\ln \frac{1+x}{1-x} = 2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}.$$
Use this to find an approximate value of $\ln 2$.
2. Find the Taylor expansion (all terms) for small $|x|$ around $x = 0$ of
$$f(x) = \frac{1}{1+x^n}, \quad n \geq 1 \text{ integer and arbitrary.}$$

8.2.2 Limiting behaviour of functions

Expand the following functions around $x = 0$ (first two or three terms), sketch them in the vicinity of $x = 0$, and find $\lim_{x\to 0} f(x)$ (can be infinity in some cases):

1. $f(x) = \frac{\sin(x)}{x}$.
2. $f(x) = \frac{\sin^2(x)}{x}$.
3. $f(x) = \frac{\cos(x)}{x}$.
4. $f(x) = \frac{x}{1+x}$.
5. $f(x) = \frac{1}{e^{\beta x}-1}, \beta > 0$ (Bose distribution).
6. $f(x) = \frac{x}{\sin(x)}$.

8.3 Physics

8.3.1 Black Body Radiation

Show that from Planck's law,

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{\exp\left(\frac{h\nu}{k_B T}\right) - 1},$$

the Wien law

$$u(\nu, T) = \frac{4\nu^3}{c^3} b \exp\left(-\frac{a\nu}{T}\right), \quad a, b = const.$$

and the Rayleigh–Jeans law

$$u(\nu, T) = \rho(\nu)\bar{E}(\nu) = \frac{8\pi\nu^2}{c^3} k_B T,$$

follow as limiting cases. Sketch Planck distributions with different temperatures T.

8.3.2 Lennard–Jones Potential

1. Calculate the first and the second derivatives of the Lennard–Jones Potential

$$V(r) = V_0 \left[\left(\frac{a}{r}\right)^{12} - 2\left(\frac{a}{r}\right)^6 \right], \quad a > 0, \quad r > 0.$$

2. Calculate the position r_0 where the potential has its minimum, and Taylor–expand it $V(r)$ around this minimum. Sketch $V(r)$ and its 'harmonic (parabolic) approximation'.

*3. Determine the angular frequency for small oscillations of a mass m around the minimum r_0.