2.1 Physics

1. A quantum particle moving in a thin wire has a wave function

$$\Psi(x) = \frac{c}{x+i},$$

where x is the coordinate of the particle and c a positive constant. Calculate and sketch the probability function $p(x) = \Psi^*(x)\Psi(x)$ of the particle ($p(x)dx$ describes the probability to find the particle in the small interval dx around x). Where is the maximum of this probability?

$p(x) = \frac{c^2}{(x-i)(x+i)} = \frac{c^2}{x^2+1}$, which is a 'Lorentzian curve', maximum at $x = 0$.

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2. Consider an AC circuit with external complex voltage $V(t) = V_0 e^{i\omega t}$ and complex current $I(t) = I_0 e^{i\omega t}$. The elements of a general circuit are resistors R, inductors L, and capacitors C. The admittance Y_R of a resistor R is defined as $Y_R = 1/R$, the admittance Y_L of an inductor L is defined as $Y_L = -i/(\omega L)$, and the admittance Y_C of a capacitor C is defined as $Y_C = i\omega C$.

a) In a circuit with R and L parallel (see figure), the total admittance Y is the sum of the two admittances, and the complex current amplitude I_0 is $I_0 = YV_0$ with V_0 being the complex voltage amplitude. Calculate the modulus $|I_0|$ of the current amplitude for this circuit.

$Y = 1/R - i/(\omega L)$ and $|I_0| = |V_0|\sqrt{(1/R)^2 + 1/(\omega L)^2}$

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b) The complex resistance of a capacitor C is $Z_C = 1/Y_C = 1/i\omega C$. The total complex resistance Z of a circuit with a resistor R and a capacitor C in series is $Z = R + Z_C$. Sketch the circuit and calculate the total complex current amplitude I_0 from $V_0 = ZI_0$, where V_0 is the total complex voltage amplitude. Then calculate the complex voltage drop V_C at the capacitor and its modulus $|V_C|$. Identify a characteristic time-scale of this circuit.

$V_0 = (R + 1/i\omega C)I_0$. Now, $V_C = Z_C I_0$ because the current I flows through the capacitor. this means $V_C = Z_C V_0/(R + 1/i\omega C) = V_0/(1 + i\omega RC)$ and $|V_C| = |V_0|/\sqrt{1 + (\omega RC)^2}$. ωRC must be dimensionless, therefore $\tau = RC$ is a characteristic time–scale of this circuit.

2.2 Math Practise

2.2.1 Basic Operations

Calculate the following:

a) $(-1+2i)(4-i)$

$= -4 + i + 8i - 2i^2 = -4 - 2(-1) + 9i = -2 + 9i$

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b) $(2-i)(-7+22i)$

$= -14 + 44i + 7i - 22i^2 = 8 + 51i$

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c) $\frac{3-2i}{-1+i}$

$= \frac{3-2i}{-1+i}\frac{-1-i}{-1-i} = \frac{-5-i}{2} = -\frac{5}{2} - \frac{1}{2}i.$

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d) i^4

$= (i^2)^2 = (-1)^2 = 1.$

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e) i^{30}

$= (i^2)^{15} = (-1)^{15} = -1.$

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*f) Find real numbers x and y such that $3x + 2iy - ix + 5y = 7 + 5i$.

Write $3x+5y+i(2y-x) = 7+5i$. Equate real and imaginary parts, $3x+5y = 7$ and $2y - x = 5$. Solve these simultaneously, $x = -1$, $y = 2$.

2.2.2 Quadratic Equations, Real and Imaginary Part

a) Find the complex solutions of $z^2 - 10z + 40 = 0$.

$z = \frac{10}{2} \pm \sqrt{10^2/4 - 40} = 5 \pm \sqrt{-15} = 5 \pm i\sqrt{15}.$

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b) Define the function $f(x) = [x+i]^{-1}$ for real x. Calculate and sketch $f_r(x) := Re[f(x)]$ and $f_i(x) := Im[f(x)]$.

$f(x) = \frac{1}{x+i} = \frac{x-i}{(x+i)(x-i)} = \frac{(x-i)}{x^2+1}$. From this we recognize $f_r(x) = x/(x^2+1)$, $f_i(x) = -1/(x^2+1)$.

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c) Show that for any complex number Ψ, the product $\Psi^*\Psi$ (where Ψ^* is the conjugate complex of Ψ) is real.

Write $\Psi = x + iy \rightsquigarrow \Psi^*\Psi = (x - iy)(x + iy) = x^2 + y^2$, which is real.

2.3 Math Problems

2.3.1 Vector and Polar Form

Let $z_1 = 2 + i$, $z_2 = 3 - 2i$.
a) Draw z_1 and z_2 as vectors in the complex plane. Calculate $z = z_1 + z_2$ and draw z as a vector in the complex plane.

$z = 5 - i$.

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b) Calculate $|z_1|$ and $|z_2|$ and explain their meaning.

$|z_1| = \sqrt{2^2 + 1^2} = \sqrt{5}$, $|z_2| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$. They are the lengths of the corresponding vectors.

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c) Prove graphically that $|z_1 + z_2| \leq |z_1| + |z_2|$. Confirm this by direct calculation.

The sum of the lengths of two sides of a triangle is greater than (or equal) to the length of the third side. $|z_1 + z_2| = \sqrt{5^2 + 1^2} = \sqrt{26} \leq \sqrt{5} + \sqrt{13}$.

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d) What is the polar form for the complex number $z = x + iy$, x and y real?

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\begin{aligned}
z &= x + iy = r\cos(\theta) + ir\sin(\theta) \\
x &= r\cos(\theta), \quad y = r\sin(\theta) \\
r &= \sqrt{x^2 + y^2} = \sqrt{z\bar{z}} \\
\tan(\theta) &= \frac{y}{x} \rightsquigarrow \theta = \arctan\left(\frac{y}{x}\right).
\end{aligned}
$$

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e) Express $z = 1 + i$ in polar form. Check the result by drawing z as a vector in the complex plane.

$r = |z| = \sqrt{2}$, $\theta = \arcsin(y/r) = \arcsin(1/\sqrt{2}) = \pi/4$.

f) Sketch the region of complex numbers in the complex plane (x–y plane) with $1 < |z| < 2$.

The inner of the ring around the origin with small radius 1 and large radius 2.

2.3.2 De Moivre's Theorem

a) Take the square of a complex number z in polar form and prove two identities for trigonometric functions from that.

$$\begin{aligned}
[\cos(\theta) + i\sin(\theta)]^2 &= [\cos(2\theta) + i\sin(2\theta)] \Leftrightarrow \\
\cos^2(\theta) - \sin^2(\theta) + 2i\cos(\theta)\sin(\theta) &= \cos(2\theta) + i\sin(2\theta) \Rightarrow \\
\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \quad &, \quad \sin(2\theta) = 2\cos(\theta)\sin(\theta).
\end{aligned}$$