Boson Cavities: From Electronic Transport to Quantum Chaos

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Abstract. This is a 'mini-review' of some recent results on electron transport through two-level systems (e.g., double quantum dots) and simple mesoscopic scatterers (delta barrier), interacting with dissipative boson baths and single boson modes (phonons, photons). The relevant models (Spin-Boson system, Rabi-Hamiltonian) and their stationary properties (electron current, boson distribution) are investigated. For a single boson mode interacting with N two-level systems, the relation between quantum chaos and a quantum phase transition for $N \to \infty$ is discussed.

A large part of solid state physics deals with the interactions between fermions (electrons) and bosons (phonons, photons, magnons etc.). There is a recent trend towards studying these interactions in their 'purest form', i.e. in quantum systems with only very few effective degrees of freedom, to be controlled from the 'outside' by external parameters such as magnetic fields or gate voltages. Typically, nanoscales and low temperatures are required in order to master the complexity of condensed matter (many-body effects, decoherence) if one wishes to realise quantum optical effects in quantum transport, or to achieve a control over the two key elements of quantum mechanics (quantum superpositions and entanglement) in an 'artificial', man-made system.

On the theoretical side, simple models for fermion-boson interactions continue to be fascinating, as often very non-trivial results can be obtained from even the most primitive Hamiltonians.

In this short overview, I discuss models that describe electron transport and two-level systems interacting with dissipative boson baths and single boson modes. The main focus will be on electron-phonon interactions, motivated by recent experiments in coupled quantum dots [1–9], 'recoil' effects in free-standing semiconductor quantum dots [10], or systems where phonons start to become controllable (phonon confinement, 'phonon cavity QED' [11–18]). More or less closely related (although not reviewed here) are situations where vibrational degrees of freedom play a big role, such as in experiments on transport through single molecules [19–24], electron 'shuttles', freestanding and movable nanostructures [25–32], or theories dealing with macroscopic 'quantum *mechanics*' of, e.g., cantilevers coupled to Cooper pair boxes [33].

1 Phonon Cavities

The simplest model for a phonon cavity is an infinitely extended, homogeneous thin plate of thickness $2b$, where phonons are described by a displacement field $\mathbf{u}(\mathbf{r})$, disregarding the microscopic crystal structure. The interactions of dilatational and flexural phonon modes with quantum dot electrons have been investigated by S. Debald *et al.* [18]. The determination of the phonon-subband dispersion relation $\omega_{n,q_\parallel}$ ($\mathbf{q}_\parallel$ is the in-plane phonon wave vector) from the Rayleigh-Lamb equations is a well-known problem from elasticity theory [34–36], although non-trivial due to the boundary conditions at the surfaces that mix longitudinal and transversal propagation.

The numerically determined $\omega_{n,q_\parallel}$ curves have minima at values for $\mathbf{q}_\parallel$ that correspond to van Hove singularities in the phonon density of states at certain finite energies $\hbar\omega$. These singularities (which are a geometrical effect and not due to, e.g., the crystal structure) are 'nanomechanical fingerprints' of confinement and lead to a strong increase in electron-phonon scattering at the corresponding energies. This has been predicted to be observable in energy-dependent non-linear transport spectroscopy in coupled quantum dots.

Another surprising feature of the thin plate model is the *vanishing* of the deformation potential (DP) interaction between electrons and phonons for $\mathbf{q}_\parallel = q_{t,n}$, where the $q_{t,n}$ denote the solutions for the transversal wavevectors associated with the transversal speed of sound c_t. For dilatational modes, the displacement field has zero divergence at the phonon energy

$$\hbar\omega_0 = \frac{\pi}{\sqrt{2}}\frac{\hbar c_t}{b}, \tag{1}$$

and the electron-phonon DP vanishes at this energy. If electrons are confined symmetrically in the midplane between the two plate surfaces, flexural phonon modes are decoupled due to symmetry. Then, the dominant second order DP electron-phonon scattering is completely 'switched off' for energy transfers $\Delta = \hbar\omega_0$ (although contributions from fourth and higher order processes still remain possible). Since piezoelectric coupling is weaker than DP interaction for small b, the $\Delta = \hbar\omega_0$ defines a 'dissipation-free manifold' for midplane electrons in, e.g., double quantum dots (see below).

It should be emphasised that in contrast to phononic crystals, the vanishing of the electron-phonon interaction and the van Hove singularities discussed here occur in a simple homogeneous, infinitely extended plate that is confined in just one direction.

2 Non-linear Transport: Dissipative Spin-Boson System

Double quantum dots are sensitive phonon emitters and detectors[3,4,37] and can be described by a (pseudo) spin-boson model [38]

$$H = \frac{\varepsilon}{2}\sigma_z + T_c\sigma_x + \frac{1}{2}\sigma_z A + \sum_{\mathbf{Q}} \omega_Q a^\dagger_{\mathbf{Q}} a_{\mathbf{Q}}, \quad A := \sum_{\mathbf{Q}} g_{\mathbf{Q}} \left(a_{-\mathbf{Q}} + a^\dagger_{\mathbf{Q}}\right), \tag{2}$$

where one additional 'transport' electron tunnels between a left (L) and a right (R) dot with energy difference ε and inter–dot coupling T_c, where $\sigma_z = |L\rangle\langle L| - |R\rangle\langle R|$ and $\sigma_x = |L\rangle\langle R| + |R\rangle\langle L|$. Here, ω_Q are the frequencies of phonons, and the g_Q denote interaction constants. The coupling to external leads offers the possibility to study spin-boson dynamics in transport properties such as the (non-)stationary electronic current or shot noise.

The simplest description is that for *non-linear transport* with the lead chemical potentials $\mu_L \to \infty$ and $\mu_R \to -\infty$ [39–41,38], allowing for an additional 'empty' state and tunneling from a left reservoir at rate Γ_L into the left dot, and from the right dot to the right reservoir at rate Γ_R. Lowest order perturbation theory in these rates (neglecting higher order terms in $\Gamma_{L/R}$ [42,43]) yields an equation of motion for the reduced statistical operator $\rho(t)$, [38,39]

$$\begin{aligned}
\frac{\partial}{\partial t}\rho_{LL}(t) &= -iT_c\left[\rho_{LR}(t) - \rho_{RL}(t)\right] + \Gamma_L\left[1 - \rho_{LL}(t) - \rho_{RR}(t)\right] \\
\frac{\partial}{\partial t}\rho_{RR}(t) &= -iT_c\left[\rho_{RL}(t) - \rho_{LR}(t)\right] - \Gamma_R\rho_{RR}(t).
\end{aligned} \tag{3}$$

For the remaining equation for the off-diagonal element $\rho_{LR} = \rho^*_{RL}$, one has to choose between perturbation theory in g_Q (weak coupling, PER), or in T_c in a polaron-transformed frame (strong coupling, POL) [44]. In general, no exact solution of the model is available even for the simplest case of only one bosonic mode (see below).

The standard Born and Markov approximation with respect to A yields

$$\begin{aligned}
\frac{d}{dt}\rho^{\mathrm{PER}}_{LR}(t) = {} & \left[i\varepsilon - \gamma_p - \Gamma_R/2\right]\rho_{LR}(t) \\
& + \left[iT_c - \delta_-\right]\rho_{RR}(t) - \left[iT_c - \delta_+\right]\rho_{LL}(t).
\end{aligned} \tag{4}$$

Here, the rates are

$$\begin{aligned}
\gamma_p &\equiv 2\pi\frac{T_c^2}{\Delta^2}\rho(\Delta)\coth\left(\beta\Delta/2\right), \quad \rho(\omega) \equiv \sum_{\mathbf{Q}} |g_{\mathbf{Q}}|^2\delta(\omega - \omega_Q) \\
\delta_\pm &\equiv -\frac{\varepsilon T_c}{\Delta^2}\frac{\pi}{2}\rho(\Delta)\coth\left(\beta\Delta/2\right) \mp \frac{T_c}{\Delta}\frac{\pi}{2}\rho(\Delta),
\end{aligned} \tag{5}$$

where $\Delta := \sqrt{\varepsilon^2 + 4T_c^2}$ is the energy difference of the hybridized levels, and $\beta = 1/k_BT$ the inverse phonon equilibrium bath temperature. Note that beside the off–diagonal decoherence rate γ_p, there appear terms $\propto \delta_\pm$ in the diagonals which turn out to be important for the stationary current.

On the other hand, the polaron transformation [38] leads to an integral equation

$$\rho_{LR}^{POL}(t) = -\int_0^t dt' e^{i\varepsilon(t-t')} \left[\frac{\Gamma_R}{2} C(t-t')\rho_{LR}(t') \right.$$
$$\left. + iT_c \left\{ C(t-t')\rho_{LL}(t') - C^*(t-t')\rho_{RR}(t') \right\} \right], \quad (6)$$

where $C(t) = \langle X_t X^\dagger \rangle$,

$$C(t) := \exp\left\{ -\int_0^\infty d\omega \frac{\rho(\omega)}{\omega^2} \left[(1-\cos\omega t)\coth(\beta\omega/2) + i\sin\omega t \right] \right\}, \quad (7)$$

is the phonon equilibrium correlation function of the displacement operators X (X_t is the time evolution of X with respect to the phonon system),

$$X = \Pi_{\mathbf{Q}} D_{\mathbf{Q}} \left(\frac{g_{\mathbf{Q}}}{\omega_Q} \right), \quad D_{\mathbf{Q}}(z) = e^{z a_{\mathbf{Q}}^\dagger - z^* a_{\mathbf{Q}}}, \quad (8)$$

where $D_{\mathbf{Q}}(z)$ is the unitary displacement operators for the phonon mode $\mathbf{Q}$.

2.1 Dissipative Landau-Zener Problem and Quantum Pump

For $\Gamma_L = \Gamma_R = 0$, one can study adiabatic transfer [45–48] of electrons from, e.g., the left to the right dot under the influence of a Hamiltonian with a slow time-dependence $T_c(t) = -\frac{\Delta}{2}\sin\Omega t$, $\varepsilon(t) = -\Delta\cos\Omega t$. This is relevant, e.g., for adiabatic quantum computation schemes in solids [49–51], where one must be fast enough in order to avoid dissipation, and slow enough in order to avoid undesired Landau-Zener transitions to excited states. This trade-off can be quantified [52,53] by calculating the inversion change $\delta\langle\sigma_z\rangle_f$ from the ideal value $\langle\sigma_z\rangle_f = -1$, which for a slow half-period sweep (duration π/Ω) yields

$$\delta\langle\sigma_z\rangle_f \approx 1 - \left[\left(\frac{\Delta}{\omega_R}\right)^2 + \left(\frac{\Omega}{\omega_R}\right)^2 \cos\left(\frac{\pi\omega_R}{\Omega}\right) \right] + 2\frac{c}{\Omega}\rho(\Delta)n_B(\Delta), \quad (9)$$

where $\omega_R \equiv \sqrt{\Omega^2 + \Delta^2}$, n_B is the Bose distribution, and $c = \frac{\pi^3 J_{3/2}(\pi)}{4\sqrt{2}} = 2.4674$. The ground state of the system 'rotates' on a curve in ε-T_c-space with constant energy difference Δ to the excited level, such that for $\Delta = \hbar\omega_0$ dissipation due to phonon absorption can be switched off in a thin plate cavity as discussed above.

The effect of dissipation on adiabatic rotation of quantum states [54,53] can in principle be measured as a time-averaged current in a 'quantum pump':

One pump cycle starts with an additional electron in the left dot and an adiabatic rotation of the parameters $(\varepsilon(t), T_c(t))$ by changing, e.g., gate voltages as a function of time. This completely quantum-mechanical part of the cycle is performed in the 'save haven' of the Coulomb- and the Pauli-blockade [55], i.e., with the left and right energy levels of the two dots well below the chemical potentials of the leads. The cycle continues with closed tunnel barrier $T_c = 0$ and increasing $\varepsilon_R(t)$; the two dots then are still in a superposition of the left and the right state. The subsequent lifting of the right level above the chemical potential of the right lead constitutes a measurement of that superposition: the electron is either in the right dot (with a high probability $1 - \frac{1}{2}\delta\langle\sigma_z\rangle_f$) and tunnels out, or the electron is in the left dot (and nothing happens because the left level is still below μ and the system is Coulomb blocked). For $\Gamma_R, \Gamma_L \gg t_{\rm cycle}^{-1}$ the decharging of the right dot and the recharging of the left dot from the left lead is fast enough to bring the system back into its initial state with one additional electron on the left dot, and the average electron current is

$$\langle I\rangle_{\rm pump} = -e\left[1 - \frac{1}{2}\delta\langle\sigma_z\rangle_f\right]/t_{\rm cycle} \tag{10}$$

Here, the leads act as classical measurement devices of the quantum-mechanical time-evolution between the two dots. Note that the present scheme combines the 'classical' pumping aspect in Coulomb blockaded systems [56–59] with tunneling/quantum interference in mesoscopic pumping [60–65]. Similar schemes for adiabatic transfer have been suggested by Silvestrini and Stodolsky [66], Barnes and Milburn [67], and realised experimentally in a superconducting Cooper pair box [68].

2.2 Stationary Current

The stationary electron current $I_{\rm stat} = -e2T_c{\rm Im}\hat{\rho}_{LR}(z=0)$ through the double quantum dot is obtained from Laplace transforming the equations of motion as an infinite sum of contributions G_+ ($= I_{\rm stat}/e$ in lowest order in T_c) and G_-,

$$I_{\rm stat}(\varepsilon) = \frac{-e\Gamma_L\Gamma_R G_+(\varepsilon)}{\Gamma_L G_-(\varepsilon) + (\Gamma_L+\Gamma_R)G_+(\varepsilon) - \Gamma_L\Gamma_R}. \tag{11}$$

Here, the expressions

$$\begin{aligned} G_\pm^{\rm (PER)} &\equiv 2T_c{\rm Im}\frac{iT_c - \delta_\pm}{i\varepsilon - \gamma_p - \Gamma_R/2} \\ \begin{matrix} G_+^{\rm (POL)} \\ G_-^{\rm (POL)} \end{matrix} &\equiv 2T_c{\rm Im}\frac{-iT_c}{1+\frac{1}{2}\Gamma_R C_\varepsilon}\cdot\begin{cases} C_\varepsilon \\ C_{-\varepsilon}^* \end{cases} \end{aligned} \tag{12}$$

are obtained in perturbation theory in g_Q (PER) and from the polaron transformation (POL, perturbation theory in T_c) and the subsequent decoupling of the correlation function $C(t)$, Eq. (7) with Laplace transform

$$C_\varepsilon := \int_0^\infty dt e^{i\varepsilon t} C(t). \tag{13}$$

Note that PER works in the correct eigenstate base of the hybridized system (level splitting Δ), whereas the energy scale ε in POL is that of the two isolated dots ($T_c = 0$) and therefore does not incorporate the square-root hybridization form of $\Delta = \sqrt{\varepsilon^2 + 4T_c^2}$. However, for large $|\varepsilon| \gg T_c$, $\Delta \to |\varepsilon|$, and POL and PER coincide: in this limit and for small electron-phonon coupling,

$$G_\pm(\varepsilon) \to -2T_c^2 \frac{\Gamma_R/2 + \gamma(\pm\varepsilon)}{\Gamma_R^2/4 + \varepsilon^2}, \gamma(\varepsilon) := \frac{\pi\rho(|\varepsilon|)}{2\varepsilon^2}\left[\coth(\beta|\varepsilon|/2) + \mathrm{sgn}(\varepsilon)\right] \tag{14}$$

For $\varepsilon \gg T_c$, the stationary current Eq.(11) is determined by the function $\gamma(\varepsilon)$, showing the broad 'shoulder' on the spontaneous phonon emission side of the resonant tunneling peak, as observed by Fujisawa *et al.* [3].

3 Non-linear Transport: Single Boson Mode

The *Rabi Hamiltonian* [69,70] is given by the single boson mode version $\omega_Q = \omega\delta_{\mathbf{Q},\mathbf{Q}_0}$, $g_Q = g\delta_{\mathbf{Q},\mathbf{Q}_0}$, of Eq. (2) or canonically equivalent forms of it. It is probably one of the best studied models for the interaction of matter with light [70] and can be used, e.g., to study the transfer of quantum coherence from light to matter (control of tunneling by electromagnetic fields [71,54]) and vice versa [72–74].

Non-equilibrium physics of single or few vibration modes in molecular electronic transport has already been studied experimentally and theoretically. Phonon cavity experiments with quantum dots in free-standing semiconductor structures give indications of 'recoil' effects related to confined phonon modes [10]. One could furthermore envisage transport experiments through *coupled* quantum dots interacting with single phonon or photon cavity modes, or single vibrational degrees of freedoms of macroscopic mechanical devices (such as cantilevers) coupled to microscopic charges. [75,33].

The Rabi Hamiltonian is probably *the* simplest model for the interaction of light with matter, and yet it is only exactly solvable at certain values of the coupling constant g ('Juddian points'), or in the rotating wave approximation. We have therefore numerically solved the master equation for non-linear transport through the coupled single mode (pseudo) spin-boson system (again corresponding to, e.g., double quantum dots in the Coulomb blockade regime), without invoking any further approximations such as decoupling schemes or perturbation theory in g. However, the bosonic Hilbert space has to be truncated at a finite number N of boson states. The total number of equations for the stationary dot-boson density operator

$$\rho_{nm}^{ij} := \langle n, i|\rho|j, m\rangle, \quad i,j = 0, L, R, \tag{15}$$

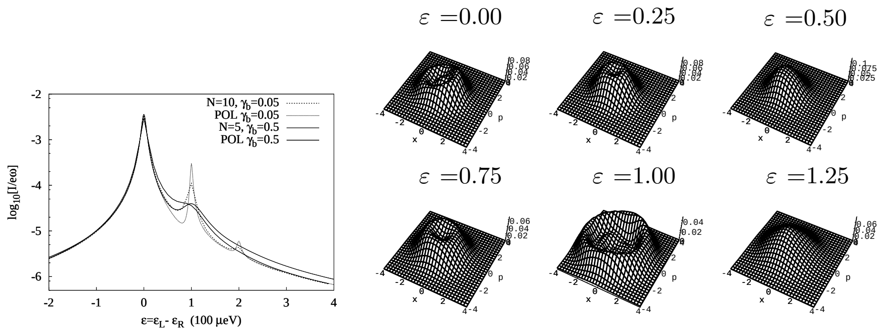

Fig. 1. Left: Stationary current through double quantum dot coupled to single boson mode (coupling constant $g = 0.2$) with tunnel rates to left/right leads $\Gamma_L = \Gamma_R = 0.1$, interdot coupling $T_c = 0.01$, boson damping $\gamma_b = 0.05$ (in units of boson frequency ω). Numerical results truncation for N boson states, alongside corresponding polaron transformation result. Right: Wigner distribution functions for the bosonic mode ($\Gamma_L = \Gamma_R = T_c = 0.1$, $\gamma_b = 0.005$, $g = 0.8$, $N = 20$.)

then is $5N^2 + 10N + 5$, remembering that there is always an equation for $n = m = 0$.

3.1 Boson Damping and Boson Distribution

Damping of the boson mode $a^\dagger$ can be described within the master equation approach in the standard Lindblad form [76],

$$\dot{\rho}|_{\text{damping}} = -\frac{\gamma_b}{2}\left(2a\rho_{\rm B}a^\dagger - a^\dagger a\rho_{\rm B} - \rho_{\rm B}a^\dagger a\right), \tag{16}$$

which can easily be incorporated either in the numerical approach, or used in the analytical expression for the stationary current $I_{\text{stat}}(\varepsilon)$, Eq. (11), from the polaron transformation and subsequent decoupling of the boson degree of freedom with the correlation function

$$C(t) = \exp\left\{-\frac{g^2}{\omega^2}\left(1 - e^{-\left(\frac{\gamma_b}{2}+i\omega\right)t}\right)\right\}, \quad t \geq 0. \tag{17}$$

The analytical result Eq. (11), together with Eqs. (12), (13), and (17) agrees relatively well with the numerical solution; both showing phonon emission peaks at $\varepsilon = n\omega$ for small T_c and small damping γ_b, cf. Fig. 1. There is, however, a more fundamental issue with the description of a damped oscillator mode by the simple Lindblad form Eq.(16), which fails [77], e.g., to reproduce power-law tails in correlation functions at low temperatures as obtained from exact solutions of microscopic models [78]. These models can be incorporated into single electron-boson transport theories [24], although decoupling approximations have to be invoked there, too.

One of the interesting question about 'molecular transport' through this spin-boson system is how the boson state can be controlled by ε, T_c, and

$\Gamma_{L/R}$, i.e., parameters of the electronic subsystem. The somehow intuitive picture of 'controlling' the boson mode by the stationary electron current is not appropriate here, because the coupled electron and boson have to be dealt with on equal footing. In fact, we have not been able to find simple limiting analytical solutions for the stationary reduced density operator of the boson,

$$\rho_b \equiv \lim_{t\to\infty} \mathrm{Tr}_{\mathrm{dot}}\rho(t) = \lim_{t\to\infty} \sum_{i=0,L,R} \rho^{ii}(t). \tag{18}$$

except for $\varepsilon \ll -|T_c|$, where $\langle \sigma_z \rangle \approx 1$ ('electron on left dot') and the boson is in the coherent state $|z = -g/2\omega\rangle$. The boson state can be visualised from the numerical result, using the Wigner function [79]

$$W(x,p) = \frac{1}{\pi} \sum_{n,m=0}^{\infty} (-1)^n \langle n|\rho_b|m\rangle\langle m|D(2\alpha)|n\rangle, \tag{19}$$

where $D(z) = \exp[za^\dagger - z^* a]$ and $\alpha \equiv (x+ip)/\sqrt{2}$. Fig. 1 (right) shows that the distribution in phase space spreads out close to the resonance energies, with corresponding peaks in position (x) and momentum (p) variances (not shown here).

4 Single Particle Scattering in a Cavity

I now turn from single electron tunneling ('zero-d transport') to mesoscopic transport in 1d systems (quantum wires), which leads me to quantum mechanical *scattering* and the transmission properties of particles in the presence of coupling to a cavity boson mode. Even for non-interacting fermions at finite density this is a very complex problem due to the possibility of induced many-body effects (Kondo physics, superconducting correlations,...) in the presence of a Fermi sea. We have therefore started with re-considering the simplest model for 1d single electron *scattering* in a boson cavity;

$$H = \frac{p^2}{2m} + \delta(x)\left\{g_0 + g_1[a^\dagger + a]\right\} + \Omega a^\dagger a. \tag{20}$$

The electron-boson coupling is via a 'dynamical' delta-barrier. Although at first sight this model might seem a bit too simple in order to yield interesting physics, quite a few authors have actually investigated this Hamiltonian or its lattice version in order to study tunneling in presence of phonons[80], Fano-type resonances [81,82] or the behaviour of transmission amplitudes in the complex energy plane [83,84], and time-dependent Hamiltonians as classical limits of fully quantised models [85].

4.1 Recursion Scheme

Scattering states of the Hamiltonian can be written as highly entangled wave functions of the coupled electron-boson system, $\langle x|\Psi\rangle = \sum_{n=0}^{\infty} \psi_n(x)|n\rangle$, where $\{|n\rangle\}$ is the harmonic oscillator basis. The total transmission coefficient $T(E)$ is obtained from the sum over all *propagating* modes,

$$T(E) = \sum_{n=0}^{[E/\Omega]} \frac{k_n(E)}{k_0(E)} |t_n(E)|^2, \quad k_n \equiv \sqrt{E - n\Omega} \tag{21}$$

where $\hbar = 2m = 1$ and the sum runs up to the largest n such that k_n remains real. Continuity of the wave function at $x = 0$ leads to an infinite recursion relation for a self energy $\Sigma^{(N)}(E)$ which can be written in an intuitive form that, e.g., for the zero-channel transmission amplitude $t_0(E)$ reads

$$t_0(E) = \frac{-2i\gamma_0(E)}{G_0^{-1}(E) - \Sigma^{(1)}(E)}, \Sigma^{(N)}(E) = \frac{Ng_1^2}{G_0^{-1}(E - N\Omega) - \Sigma^{(N+1)}(E)} \tag{22}$$

where the 'Greens function' $G_0(E) \equiv [-2i\gamma_0(E) + g_0]^{-1}$ and $\gamma_0(E) = \sqrt{E}\theta(E) + i\sqrt{-E}\theta(-E)$.

The recursion can be truncated by setting $\Sigma^{(N)}(E) = 0$ for a fixed $N > 0$ and recursively solving Eq. (22) down to $\Sigma^{(1)}(E)$ which, however, fails to work for too large coupling constants g_1. In the strong coupling regime, one has to start from polarons as new quasi-particles. In a lattice version of the Hamiltonian Eq. (20), this is easily accomplished by a canonical transformation and a subsequent perturbation theory in the coupling to the localised polaron level, similar to the spin-boson problem discussed above. On the other hand, in the original Hamiltonian Eq. (20) this would correspond to an inconvenient perturbation theory in the *kinetic* energy $p^2/2m$ of the electron.

4.2 Resonances and Transparency Points

Barriers with an attractive static part, $g_0 < 0$, are most interesting in that they exhibit Fano type resonances with zero transmission coefficient below the first threshold $E = \Omega$. If the term $(a^\dagger + a)$ in the Hamiltonian Eq. (20) is replaced by an oscillating term $\propto \cos(\Omega t)$, Fano-resonances in this classical limit are known to appear [82] when the energy of the electron in the first (non-propagating) channel $n = 1$ coincides with the bound state of the attractive delta barrier potential, $E - \Omega = -g_0^2/4$. In the single boson mode case Eq. (20), zero transmission corresponds to a diverging self energy $\Sigma^{(1)}(E)$, in Eq. (22). For small g_1, this condition can be written as

$$0 = [\Sigma^{(1)}(E)]^{-1} \approx (2\sqrt{\Omega - E} + g_0)/g_1^2, \tag{23}$$

which coincides with the resonance condition for the classical case.

A new feature of Eq. (20) (which does not appear for its classical, time-dependent counterpart) is the existence of perfect transmission $T(E) = 1$ at an energy below the opening of the first channel ($n = 1$) threshold. There, $t_0(E) = -2ik_0/(-2ik_0 + g_0 - \Sigma^{(1)}(E))$ which means that $t_0(E) = 1$ for $g_0 = \Sigma^{(1)}(E)$. Since $\Sigma^{(1)}(E)$ is real for $0 < E < \Omega$, the self energy exactly renormalizes the static part g_0 of the scattering potential to zero at this point. Using the perturbative expression in Eq.(23) for $\Sigma^{(1)}(E)$, one finds two perfect transparency points

$$g_0 = -\sqrt{\Omega - E} \pm \sqrt{\Omega - E + g_1^2}, \tag{24}$$

i.e, both for attractive and repulsive static barrier strengths g_0.

5 Quantum Chaos and Quantum Phase Transition: Single Mode Dicke Model

In this final section, I return back to the single-mode (pseudo) spin boson problem but now consider not only a single two-level system (as represented by the Pauli matrices σ_i), but an array of $N = 2j$ identical (but distinguishable) two-level systems represented by angular momentum operators J_z, $J_\pm$ for a pseudo-spin of length j. The corresponding Dicke Hamiltonian [86]

$$H = \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{2j}}(a^\dagger + a)(J_+ + J_-), \tag{25}$$

generalises the Rabi Hamiltonian to $j > 1/2$, is well-known from quantum optics (single mode superradiance) and can be regarded as a simple model for solid state qubit arrays interacting via a common cavity boson (photon/phonon).

Our original motivation to study this model was to find a relation between quantum chaos (showing up for finite N) and quantum phase transitions ($N \to \infty$) in systems of N interacting particles as a function of some coupling constant λ. For non-interacting systems, the Anderson (localisation-delocalisation) transition is an example for such a relation, other examples include chaos in interacting spin systems [87], the Lipkin model [88], and the interacting boson model [89].

A common feature of the models in the previous sections is the difficulty to continuously move from their weak-coupling to their strong-coupling limits. In fact, a typical feature of models like the (isolated) Rabi-Hamiltonian ($j = 1/2$) is the breakdown of numerical approaches for too large coupling constants (coming from the weak coupling side). The idea therefore is that such instabilities have a 'deeper' reason, i.e., an underlying quantum phase transition that only becomes apparent if the system is regarded as a finite-size version of some 'larger' system in the thermodynamic limit. In the example here, the Rabi Hamiltonian is the finite size version of the Dicke Hamiltonian.

We have proven and studied this connection [90,91] in great detail for Eq.(25). One might speculate that similar strong connections between quantum chaos and quantum phase transitions are a general feature of many more classes of physically interesting systems.

5.1 Spectrum and Wave Functions

We have derived [90] exact analytical solutions for the spectrum and wave functions of this Hamiltonian for $N \to \infty$ and found a localisation-delocalisation transition in a cross-over between Poissonian and Wigner level-spacing distribution, using numerical diagonalisation for finite N. The ground state bifurcates into a Schrödinger cat above the critical point $\lambda = \lambda_c = \sqrt{\omega\omega_0}/2$, which can be related to a transition between non-chaos and chaos in the classical, canonical limit of the Hamiltonian Eq.(25) and its non-linear, momentum-dependent potential energy.

A Holstein-Primakoff transformation [92] of the Hamiltonian leads to a representation in terms of two oscillator modes $a^\dagger$ and $b^\dagger$ (the latter represents the pseudo spin). For $N \to \infty$, the ground state energy

$$E_G/j = \begin{cases} -\omega_0, & \lambda < \lambda_c \\ -\left[\frac{\lambda^2}{\omega} + \frac{\omega_0^2\omega}{8\lambda^2}\right], & \lambda > \lambda_c \end{cases} \tag{26}$$

and the two collective excitation energies $\varepsilon_\pm$ are obtained exactly. From the vanishing of ε_- at the critical point one obtains the critical exponents $\nu = 1/4, z = 2$ on resonance $\omega = \omega_0$.

Finite-j precursors of the phase transitions can be identified in the crossover of the level spacing distribution $P(S)$ from Poissonian ($\lambda < \lambda_c$) to Wigner-Dyson, which we have calculated numerically [90]. The ground state wave function Ψ_0 can be represented in a 2d position space with coordinates $x \equiv (1/\sqrt{2\omega})(a^\dagger + a)$ and $y \equiv (1/\sqrt{2\omega_0})(b^\dagger + b)$. The splitting of Ψ_0 into a superposition of two peaks (separated by a distance of the order $\sqrt{j}$) is related to the existence of a conserved parity $\Pi = \exp\{i\pi[a^\dagger a + J_z + j]\}$ of the Hamiltonian. In fact, Eq. (25) is equivalent to a single particle on a lattice with points (n,m), $|m| \le j$, $n = 0,1,2,...$, and the eigenvalues ± 1 of Π correspond to the two independent sublattices. For $j \to \infty$, the effective tunnel barrier between the two lobes of Ψ_0 becomes infinitely strong, Π is spontaneously broken (the cat is 'broken into two pieces'), and each of the two lobes aquires its own effective Hamiltonian [90] above λ_c.

5.2 Classical Limit, Chaos

The above discussion shows that the simple one-boson mode Hamiltonian Eq.(25) is an attractive model to study an exact solution for a quantum phase transition. Moreover, for finite j it exhibits a well-defined transition from integrable to chaotic behavior. One can derive a canonical, classical Hamilton function corresponding to Eq.(25) by using the Holstein-Primakoff expressions for the spin, which leads to the problem of a single particle in a momentum dependent potential

$$U(x,y,p_y) = \frac{1}{2}\left(\omega^2 x^2 + \omega_0^2 y^2\right) + 2\lambda\sqrt{\omega\omega_0}xy\sqrt{1 - \frac{\omega_0^2 y^2 + p_y^2 - \omega_0}{4j\omega_0}} \tag{27}$$

Poincaré sections for the classical model [91] show the transition between regular ($\lambda < \lambda_c$) and chaotic ($\lambda > \lambda_c$) behavior which agrees with the transition in $P(S)$ of the quantum model.

6 Conclusions

In the above overview, I have presented spin-boson models from a 'mesoscopic' point of view. The coupling to external electron reservoirs allows to study these models under transport (non-equilibrium) conditions. In spite of their simplicity, the Hamiltonians discussed here have some very non-trivial properties that became apparent in particular in the last two sections (mesoscopic 'quantum' scatterer, Dicke model). In the future, photon and phonon cavities can be expected to yield further insight into the dynamics of coupled quantum systems, in particular if they are combined with electron transport.

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