NAME: ____________________

To be returned until Tuesday, April 29, 10.00 h to the Departmental Office.

Answer all questions

1.1 Question: Potential Wells

Consider the motion of a particle of mass m within the finite potential well of width $2a$,

$$V(x) = \begin{cases} 0, & -\infty < x \leq -a \\ -V, & -a \leq x \leq a \\ 0 & a \leq x < \infty \end{cases}$$

Consider the general form of the anti–symmetric (odd) eigenfunctions $\psi_o(x)$ of energy $-V < E < 0$,

$$\psi_o(x) = \begin{cases} -A_1 e^{\kappa x}, & -\infty < x \leq -a \\ A_2 \sin(kx), & -a \leq x \leq a \\ A_1 e^{-\kappa x}, & a \leq x < \infty \end{cases}$$

i) How are κ and k related to the energy E and the potential strength V?

(2 MARKS)

ii) Derive the transcendental equation that relates κ with k (you do not need to solve that equation, just derive it).

(8 MARKS)

1.2 Question: Potential Scattering

We consider a one–dimensional potential step $V(x)$ and a stationary wave function $\psi(x)$ at energy E,

$$V(x) = \begin{cases} 0 & -\infty < x \leq 0 \\ -V < 0 & 0 \leq x < \infty \end{cases}$$

$$\psi(x) = \begin{cases} e^{ikx} + re^{-ikx}, & -\infty < x \leq 0 \\ te^{ik'x} & 0 \leq x < \infty \end{cases}$$

a) Consider the case $E > 0$ and show that k and k' are real wave vectors. Give the explicit form of k and k'.

1. By consideration of the boundary conditions at $x = 0$, express t and r in terms of k and k'.

2. Calculate and sketch the transmission coefficient

$$T = \frac{k'}{k}|t|^2$$

as a function of the energy E.
3. Calculate and sketch the reflection coefficient

$$R = |r|^2$$

as a function of the energy E.
4. Show that $T + R = 1$.

(15 MARKS)

b) Consider again the case $E > 0$. The current density is defined as

$$j(x) = -\frac{i\hbar}{2m}\left[\psi(x)^*\psi'(x) - \psi(x)\left(\psi'(x)\right)^*\right].$$

Calculate the current density $j(x < 0)$ on the left side and the current density $j(x > 0)$ on the right side. Show that $j(x < 0) = j(x > 0)$ (Hint: use $1 = T + R$ (where $T = (k'/k)|t|^2$ and $R = |r|^2$).

(10 MARKS)

1.3 Superposition

Consider two eigenstates $\Psi_1(x)$ (energy eigenvalue E_1) and $\Psi_2(x)$ (energy eigenvalue E_2) of a stationary Schrödinger equation. Consider the superposition at time $t = 0$,

$$\Psi(x, t = 0) = \frac{1}{\sqrt{2}}\left[\Psi_1(x) + \Psi_2(x)\right].$$

1. Write down the form of the wave function $\Psi(x, t)$ at time $t > 0$.
2. Assume $\Psi_1(x)$ and $\Psi_2(x)$ are real functions. Write down an expression for the probability density for finding the particle at position x as a function of time t.

(8 MARKS)