NAME: ____________________

To be returned until Tuesday, May 07, 10.00 h to the Departmental Office.
Answer question 2.1 or, alternatively, question 2.2.

2.1 Question: The Harmonic Oscillator (30 Marks)

We consider the one–dimensional harmonic oscillator of mass m and frequency ω. We define the ladder operators

$$a = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}, \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2m\hbar\omega}}\hat{p}.$$

A) (15 MARKS)

Express the Hamiltonian $\hat{H} = \hat{p}^2/2m + (1/2)m\omega^2 x^2$ with the help of the ladder operators. Note that the commutator $[a, a^\dagger] = 1$. What is the lowest possible eigenvalue of the energy ?

B) (7 MARKS)

The operator $a^\dagger$ is known as a 'creation' or a 'ladder–up' operator if the following relationship holds:

$$a^\dagger|n\rangle = C_n|n+1\rangle.$$

Show that the constant C_n is $C_n = \sqrt{n+1}$ (up to a phase factor). Hint: start from $n = \langle n|a^\dagger a|n\rangle$ and re–write $a^\dagger a$ as $aa^\dagger - [a, a^\dagger]$ (the commutator $[a, a^\dagger] = 1$). Comment on the term 'ladder–operator'.

C) (8 MARKS)

Use the first excited state wave function

$$\Psi_1(x) = \frac{2\alpha^{3/2}}{\sqrt{2\sqrt{\pi}}} x e^{-\alpha^2 x^2/2}, \quad \alpha := \sqrt{m\omega/\hbar},$$

and the relation $a^\dagger \Psi_n = \sqrt{n+1}\Psi_{n+1}$ to explicitely calculate the wave function of the second excited state $\Psi_2(x)$.

2.2 The Time–Dependent Schrödinger Equation (30 MARKS)

In this problem, we solve the time–dependent Schrödinger equation for the wave function $\Psi(x, t > 0)$ of a free particle in one dimension (zero potential) with the initial condition

$$\Psi(x, t = 0) = C \exp(-x^2/(2\sigma^2)).$$

by Fourier transformation. The Fourier transformation $f(k)$ of a function $f(x)$ is defined by the equations

$$f(k) \;=\; \int_{-\infty}^{\infty} dx e^{-ikx} f(x), \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ikx} f(k).$$

a) (10 MARKS) By performing a Fourier transformation with respect to the x–coordinate, write down the time–dependent Schrödinger equation (V=0) in k–space and solve the k–space equation for $\Psi(k, t)$.

b) (10 MARKS)

Perform an inverse Fourier transformation on $\Psi(k, t)$, using the relation

$$\int_{-\infty}^{\infty} dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/4a}.$$

Show that the final result takes the form

$$\Psi(x, t) \;\propto\; \exp\left(-\frac{x^2}{2\sigma^2 + 2i\hbar t/m}\right), \tag{2.1}$$

where $\propto$ means 'proportional to'.

c) (10 MARKS)

Calculate the absolute square $|\Psi(x, t)|^2$, Eq.(2.1), neglecting all prefactors. From this you can read off how the wave packet broadens as a function of time. Calculate this width as a function of time t explicitely and compare it to the original width σ^2 of the wave packet at time $t = 0$.