Non-Markovian Quantum Dynamics of Open Systems

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Contents

• Open quantum systems
• Information flow and the trace distance
• Quantum non-Markovianity
• Local detection of initial correlations
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Recent Review:
Open quantum systems

Open system $S$: density matrix $\rho_S = \text{tr}_E \rho$

Environment $E$: density matrix $\rho_E = \text{tr}_S \rho$

Total system $S+E$:

Density matrix matrix $\rho_{SE}$

Hamiltonian $H = H_S + H_E + H_I$

Applications:

- Atomic physics/quantum optics
- Condensed matter
- Quantum chemistry
- Quantum information
- Quantum thermodynamics
- Relaxation and dissipation
- Decoherence
- Quantum theory of measurement and control
- Transport processes
- …
Quantum dynamical maps

\[ \rho_{SE}(0) = \rho_S(0) \otimes \rho_E \]  
\[ \xrightarrow{\text{unitary evolution}} \]  
\[ \rho_{SE}(t) = U_t [\rho_S(0) \otimes \rho_E] U_t^\dagger \]  
\[ \xrightarrow{\text{tr}_E} \]  
\[ \rho_S(0) \]  
\[ \xrightarrow{\text{dynamical map}} \]  
\[ \rho_S(t) = \Phi_t \rho_S(0) \]  

Quantum dynamical map (quantum channel) \( \Phi_t \):

\[ \rho_S(0) \longrightarrow \rho_S(t) = \Phi_t \rho_S(0) = \text{tr}_E \left\{ U_t [\rho_S(0) \otimes \rho_E] U_t^\dagger \right\} \]

\[ \Phi_t = \text{completely positive map} \]

Quantum process: One-parameter family of maps:

\[ \{ \Phi_t \mid \Phi_0 = I, \ 0 \leq t \leq T \} \]
Examples of quantum Markov processes

Quantum dynamical semigroup:

\[ \Phi_{t_1+t_2} = \Phi_{t_2} \Phi_{t_1} \implies \Phi_t = e^{Lt} \]

with Lindblad generator:

\[ \mathcal{L}\rho_S = -i [H_S, \rho_S] + \sum_i \gamma_i \left[ A_i \rho_S A_i^\dagger - \frac{1}{2} \{ A_i^\dagger A_i, \rho_S \} \right] \quad \gamma_i \geq 0 \]

\[ \implies \text{Master equation in Lindblad form:} \]

\[ \frac{d}{dt} \rho_S(t) = \mathcal{L}\rho_S(t) \]

Microscopic derivation: \textbf{Separation of time scales:}

\[ \tau_E \ll \tau_R \]
Non-Markovian quantum dynamics

- Standard Markov condition $\tau_E \ll \tau_R$ violated, no rapid decay of environmental correlations
- Correlation functions of higher order play important role, strong memory effects
- Finite revival times (finite reservoir)
- Correlations and entanglement in the initial state

Methods:
- Projection operator techniques
- Memory kernel master equations: Nakajima, Zwanzig, Mori,…
- Time-local master equations: Shibata, van Kampen,…
- Stochastic wave function techniques
- Influence functional/Quantum Monte Carlo
- HEOM
- Multiconfiguration wave function methods
- Exact simulation techniques
- …
Non-Markovian quantum dynamics

- What is the key feature of non-Markovian dynamics?
- How can one define non-Markovianity without reference to specific representation, master equation, approximation etc.?
- Memory and information flow?
- Construction of a measure for non-Markovianity?
- Experimentally measurable quantity?
Distance between quantum states

Trace norm of an operator $A$:

$$||A|| = \text{tr}|A| \quad |A| = \sqrt{A^\dagger A}$$

For selfadjoint operators:

$$A = \sum_i a_i |i\rangle\langle i|$$

$$||A|| = \sum_i |a_i|$$

Trace distance between quantum states $\rho_1$ and $\rho_2$:

$$D(\rho_1, \rho_2) = \frac{1}{2}||\rho_1 - \rho_2|| = \frac{1}{2}\text{tr}|\rho_1 - \rho_2|$$
Properties of the trace distance

- Metric on state space with $0 \leq D(\rho_1, \rho_2) \leq 1$

  $\rho_1 = \rho_2 \iff D = 0$

  $\rho_1 \perp \rho_2 \iff D = 1$

  $\rho_i = |\psi_i\rangle\langle\psi_i| \implies D = \sqrt{1 - |\langle\psi_1|\psi_2\rangle|^2}$

- Unitary invariance, triangular inequality, subadditivity

- Representation through maximum over all projections or positive operators $\Pi \leq I$:

  $D(\rho_1, \rho_2) = \max_{\Pi} \text{tr}\left\{ \Pi(\rho_1 - \rho_2) \right\}$
Example: Qubit

Bloch sphere:

\[ \rho_{1,2} = \frac{1}{2} \left( I + \vec{v}_{1,2} \cdot \vec{\sigma} \right) \]

Bloch vectors:

\[ |\vec{v}_{1,2}| \leq 1 \]

Euclidean distance \( = 2 \cdot \text{(trace distance)} \)
Physical interpretation

Alice

Preparation:

\( \rho_1 \) or \( \rho_2 \)

with \( p_1 = p_2 = 1/2 \)

Bob

 Measurement:

\( \rho_1 \) or \( \rho_2 \)?

\[ \text{prob} = (1 + D(\rho_1, \rho_2))/2 \]

\[ \Rightarrow D(\rho_1, \rho_2) = \text{Measure for distinguishability of } \rho_1 \text{ and } \rho_2 \]

Example: \( \rho_1 \perp \rho_2 \Rightarrow \text{prob} = 1 \)
Dynamical maps are **contractions** for the trace distance:

\[ D(\Phi \rho_1, \Phi \rho_2) \leq D(\rho_1, \rho_2) \]

⇒ no trace preserving quantum operation can increase the distinguishability of two states
Dynamics of initial state pairs

\[ \rho_{S}^{1,2}(0) \rightarrow \rho_{S}^{1,2}(t) = \Phi_t \rho_{S}^{1,2}(0) \]

Trace distance evolution:

\[ D(t) = D(\rho_{S}^{1}(t), \rho_{S}^{2}(t)) \leq D(\rho_{S}^{1}(0), \rho_{S}^{2}(0)) \]

Monotonic decrease of \( D(t) \): Decrease of distinguishability:
Flow of information from system to environment

\( D(t) \) temporarily increases: Increase of distinguishability:
Flow of information from environment back to system
Definition of quantum non-Markovianity

Definition: A quantum process $\Phi_t$ is non-Markovian iff

$$\sigma(t) = \frac{d}{dt}D(\rho^1_S(t), \rho^2_S(t)) > 0$$

for some initial state pair $\rho^1_S, \rho^2_S(0)$ and some time $t > 0$

- Increase of the distinguishability of the states $\rho^1, \rho^2_S$
- Flow of information from environment back to system
- Environment acts as memory
Example: Non-Markovian decay

Spectral density:

\[ J(\omega) = \frac{\gamma_0}{2\pi} \frac{\lambda^2}{(\omega_0 - \omega)^2 + \lambda^2} \]

Spectral width:

\[ \lambda = \tau_E^{-1} \]

Relaxation rate:

\[ \gamma_0 = \tau_R^{-1} \]

Initial states:

\[ \rho_1(0) = |1\rangle\langle 1| \]
\[ \rho_2(0) = |0\rangle\langle 0| \]
**Example: Spin bath**

**Hamiltonian:**

\[ H = \frac{\omega_0}{2} \sigma_3 + \sum_{k=1}^{N} A_k \vec{\sigma} \cdot \vec{\sigma}^{(k)} \]

**Initial states:**

\[ \rho_1(0) = |\psi\rangle\langle\psi| \]

\[ |\psi\rangle = \frac{1}{\sqrt{2}} (|+\rangle + |--\rangle) \]

\[ \rho_2(0) = \frac{1}{2} I \]
Measure for non-Markovianity

\[ N(\Phi) = \max_{\rho_S^{1,2}(0)} \int_{\sigma > 0} dt \sigma(t) \quad \sigma(t) = \frac{d}{dt} D(\rho_S^1(t), \rho_S^2(t)) \]

- Measures total backflow of information
- Summation over all time intervals in which \( \sigma > 0 \)
- Maximum over all pairs of initial states \( \rho_S^{1,2}(0) \)
- \( N(\Phi) > 0 \iff \) process non-Markovian
- Possible values: \( 0 \leq N(\Phi) \leq +\infty \)

Example: Non-Markovian decay

Gray dots: Random pairs of initial states
Circles: Optimal pure initial state pair: \( (|1\rangle \pm |0\rangle) / \sqrt{2} \)
Information flow

Information inside open system:

\[ \mathcal{I}_{\text{int}}(t) = D(\rho_1^S(t), \rho_2^S(t)) \]

Information outside open system:

\[ \mathcal{I}_{\text{ext}}(t) = D(\rho_1(t), \rho_2(t)) - D(\rho_1^S(t), \rho_2^S(t)) \geq 0 \]

Information flow:

\[
\frac{d}{dt} \mathcal{I}_{\text{int}}(t) < 0 \quad \text{open system looses information (Markovian)}
\]

\[
\frac{d}{dt} \mathcal{I}_{\text{int}}(t) > 0 \quad \text{open system gains information (non-Markovian)}
\]
Conservation of information:

\[ \mathcal{I}_{\text{int}}(t) + \mathcal{I}_{\text{ext}}(t) = \mathcal{I}_{\text{int}}(0) = \text{const} \]

General inequality based on properties of the trace distance:

\[ \mathcal{I}_{\text{ext}}(t) \leq D(\rho^1(t), \rho^1_S(t) \otimes \rho^1_E(t)) + D(\rho^2(t), \rho^2_S(t) \otimes \rho^2_E(t)) + D(\rho^1_E(t), \rho^2_E(t)) \]

Interpretation: Information outside the open system implies

- system-environment correlations
- or different environmental states
Alice prepares two quantum states $\rho_S^{1,2}$ with probabilities $p_{1,2}$ which need not be equal (biased preparation of states).

The maximal probability for a successful state discrimination Bob can achieve by an optimal strategy is given by

$$P_{\text{max}} = \frac{1}{2} \{ 1 + ||\Delta|| \}$$

where $\Delta$ is the Helstrom matrix:

$$\Delta = p_1 \rho_S^1 - p_2 \rho_S^2$$

Generalized definition of non-Markovianity

Replacing the trace distance by the norm of the Helstrom matrix leads to **generalized non-Markovianity measure**:

\[
N(\Phi) = \max_{||\Delta||=1} \int_{\sigma > 0} dt \sigma(t) \quad \sigma(t) = \frac{d}{dt}||\Phi_t\Delta||
\]

where the maximum is taken over all **Helstrom matrices with unit trace norm**:

\[
\Delta = p_1\rho_S^1 - p_2\rho_S^2 \quad ||\Delta|| = 1
\]

Generalized definition of non-Markovianity

Advantages of this approach:

- **Markovianity is equivalent to P-divisibility** of dynamical maps
- **Orthogonality of optimal state pairs** and local representation
- It leads to a **general classification** of quantum processes in open systems
- It yields direct connection to notion of a **classical Markov process**
Divisibility of quantum processes

If inverse of dynamical maps exists we define for all $t \geq s \geq 0$:

$$\Phi_{t,s} = \Phi_t \Phi_s^{-1} \implies \Phi_{t,0} = \Phi_{t,s} \Phi_{s,0}$$

Definition:

$\Phi_{t,s}$ completely positive $\iff$ process is CP-divisible

\(\Phi_{t,s}\) positive $\iff$ process is P-divisible

Definition of non-Markovianity based on CP-divisibility:


For generalized definition based on Helstrom matrix:

Markovianity is equivalent to P-divisibility

Relation to time-local master equation

Most general structure of time-local master equation:

\[
\frac{d}{dt} \rho_S = \mathcal{K}_t \rho_S \\
= -i [H_S(t), \rho_S] + \sum_i \gamma_i(t) \left[ A_i(t) \rho_S A_i^\dagger(t) - \frac{1}{2} \{ A_i^\dagger(t) A_i(t), \rho_S \} \right]
\]

- Dynamics is **CP-divisible** if and only if
  \[ \gamma_i(t) \geq 0 \]

- Dynamics is **P-divisible** if and only if
  \[ W_{nm}(t) \equiv \sum_i \gamma_i(t) |\langle n | A_i(t) | m \rangle|^2 \geq 0 \]
Relation to time-local master equation

Pauli master equation in instantaneous eigenbasis of \( \rho_S(t) \):

\[
\frac{d}{dt} P_n(t) = \sum_m \left[ W_{nm}(t)P_m(t) - W_{mn}(t)P_n(t) \right]
\]

where

\[
W_{nm}(t) = \sum_i \gamma_i(t) \left| \langle n(t)|A_i(t)|m(t) \rangle \right|^2
\]

\( \implies \) can be interpreted as Chapman-Kolmogorov equation of a classical Markov process if and only if

\[
W_{nm}(t) \geq 0
\]

Conclusion:

quantum Markovian \( \iff \) P-divisible \( \iff \) classical Markovian
Local representation and classification of dynamics

- Orthogonality of optimal state pairs:
  \[ \mathcal{N}(\Phi) = \max_{p_{1,2}, \rho^1_S \perp \rho^2_S} \int_{\sigma > 0} dt \sigma(t) \]
  \[ \sigma(t) = \frac{d}{dt} ||\Phi_t\Delta|| \]

- Local representation:
  \[ \mathcal{N}(\Phi) = \max_{p_{1,2}, \rho \in \partial U(\rho_0)} \int_{\sigma > 0} dt \sigma(t) \]
  \[ \sigma(t) = \frac{d}{dt} ||\Phi_t\Delta|| / ||\Delta|| \]

- Classification of quantum processes:

<table>
<thead>
<tr>
<th>Markovian</th>
<th>non-Markovian</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{N}(\Phi) = 0 )</td>
<td>( \mathcal{N}(\Phi) &gt; 0 )</td>
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</table>

<table>
<thead>
<tr>
<th>P-divisible</th>
<th>not P-divisible</th>
</tr>
</thead>
<tbody>
<tr>
<td>CP-divisible</td>
<td>non-P-divisible</td>
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</tbody>
</table>

\( \Phi_t \) invertible
\( \Phi_t \) not invertible
\( \partial U(\rho_0) \)
\( \rho_0^* \)
\( S(\mathcal{H}) \)
• Increase of the trace distance over initial value implies initial system-environment correlations

• Local detection of correlations
Measure for correlations in total state $\rho$:

$$D(\rho, \rho_S \otimes \rho_E) = \text{distinguishability of } \rho \text{ and } \rho_S \otimes \rho_E$$

Dynamics with initial correlations:

$$D \left( \rho^1_S(t), \rho^2_S(t) \right) - D(\rho^1_S, \rho^2_S) \leq D(\rho^1_{SE}, \rho^1_S \otimes \rho^1_E) + D(\rho^2_{SE}, \rho^2_S \otimes \rho^2_E) + D(\rho^1_E, \rho^2_E)$$
Given state: $\rho$

Generate second state $\rho'$ by **local operation** $\Phi$:

$$\rho' = (\Phi \otimes I)\rho$$

**Implications:**

- Environmental states are identical:

$$\rho_E = \rho'_E$$

- No correlations are created:

$$\rho \text{ uncorrelated} \implies \rho' \text{ uncorrelated}$$
Dynamical detection of initial correlations

Change of local trace distance:

\[ D(\rho_S(t), \rho'_S(t)) - D(\rho_S(0), \rho'_S(0)) \leq D(\rho, \rho_S \otimes \rho_E) + D(\rho', \rho'_S \otimes \rho_E) \]

Consequences:

● Any increase of the trace distance over initial value implies that \( \rho \) was correlated
● Initial correlations in \( \rho \) become dynamically perceivable in the open system at later times
● General scheme does not require knowledge of total state, environment, interaction Hamiltonian...

Local detection of quantum correlations

\[ \Phi = \text{local dephasing map} \]

\[ D(\rho_S(t), \rho'_S(t)) \leq D(\rho(0), \rho'(0)) \]

\[ \Rightarrow \text{lower bound for quantum correlations (quantum discord)} \]


Summary

• Non-Markovian processes:

\[ \sigma(t, \rho_{1,2}(0)) = \frac{d}{dt} D(\rho_1(t), \rho_2(t)) > 0 \]

\[ \Rightarrow \text{Reversed flow of information from environment to system} \]

• Measure for non-Markovianity:

\[ \mathcal{N}(\Phi) = \max_{\rho_{1,2}(0)} \int_{\sigma > 0} dt \, \sigma(t, \rho_{1,2}(0)) \]

\[ \Rightarrow \text{Measure for total backflow of information} \]

• Increase of trace distance over initial value:

\[ D(\rho_S(t), \rho'_S(t)) - D(\rho_S(0), \rho'_S(0)) \leq D(\rho, \rho_S \otimes \rho_E) + D(\rho', \rho'_S \otimes \rho_E) \]

\[ \Rightarrow \text{local witness for initial quantum correlations} \]
Further applications

- Brownian motion/Optomechanical systems:
  S. Gröblacher, A. Trubarov, N. Prigge, G. D. Cole, M. Aspelmeyer, J. Eisert,
  Nat. Commun. 6, 7606 (2015)

- Chaotic systems:
  93, 022117 (2016)

- Energy transfer in photosynthetic complexes:

- Quantum metrology:

- Quantum phase transitions:
  M. Gessner, M. Ramm, H. Häffner, A. Buchleitner, HPB, EPL 107, 40005 (2014)

- Anderson localization:
  S. Lorenzo, F. Lombardo, F. Ciccarello, G. M. Palma, arXiv:1609.04158

- ...
Experiments

• Photonic systems:

• NMR:

• Trapped ions: