Chaos and phase transitions in quantum dots coupled to bosons

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Abstract. We propose a simple model describing the collective interaction of an array of N quantum dots with a single bosonic mode. This model exhibits a quantum phase transition in the thermodynamic limit $N \to \infty$, and we describe how the precursors of this transition give rise to the appearance of quantum chaos in the system.

1. Introduction

Quantum dots have been shown to exhibit a number of quantum coherent effects when coupled to each other or to external radiation. Coherent superpositions of states in single and double quantum dots [1], the spontaneous emission of phonons [2] and non-linear effects like dressed states (photo-sidebands) in dots [3] are examples of quantum-optical effects in controllable, artificial atoms [4]. Coherently coupled systems of $N \geq 2$ quantum dots with two internal (spin or orbital) degrees of freedom have been suggested as models for 'qubit' arrays in quantum computation. In such models, collective behaviour, similar to superfluorescence in atomic systems, has been predicted to show up in transport, optical, and (de–)coherence properties [5].

In the following, we seek to investigate how collective behaviour is connected to the appearance of quantum chaos in arrays of quantum dots coupled to a bosonic degree of freedom such as a photon or phonon mode. To this end, we consider the Dicke Hamiltonian [6], a simple model consisting of N identical two-level systems interacting with a single bosonic mode. This model exhibits a quantum phase transition (QPT) as a function of a coupling constant λ in the thermodynamic limit, $N \to \infty$.

Recently, we have reported on an *exact* solution for all eigenstates [7], eigenvalues and critical exponents in the thermodynamic limit, and shown that above the critical point $\lambda = \lambda_c$ the ground-state wavefunction bifurcates into a Schrödinger cat state for any $N < \infty$. In this contribution, we concentrate on the statistics of the energy spectrum and demonstrate that quantum chaos occurs and is correlated with the precursors of the QPT.

2. The Model

We model each Qdot as a two-level system, all of which have identical level–splitting $\hbar\omega_0$. This collection of N two-level systems may be described in terms of a single $(N+1)$-level system, provided that we treat all Qdot configurations with the same number of excitations as identical, i.e. we ignore the spatial distribution of the Qdots. The resulting $(N+1)$-level system may then be pictured as a pseudo-spin vector of length $j = \frac{N}{2}$, described by the angular momentum operators, $\{J_i;\ i = z, \pm\}$, which obey the usual commutation relations

	$\lambda < \lambda_c$	$\lambda > \lambda_c$
E_G/j	$-\omega_0$	$-\frac{2\lambda^2}{\omega} + \frac{\omega_0^2\omega}{8\lambda^2}$
$< J_z > /j$	-1	$-\frac{\omega\omega_0}{4\lambda^2}$
$< a^\dagger a > /j$	0	$\frac{16\lambda^4 - \omega_0^2\omega^2}{8\omega^2\lambda^2}$

Table 1. The energy, inversion and field occupancy of the the ground state of Hamiltonian (1) above and below the phase transition for the system in the thermodynamic limit.

$[J_z, J_\pm] = \pm J_\pm$ and $[J_+, J_-] = 2J_z$. We then let this Qdot system interact with a single bosonic mode of frequency ω via a dipole–type interaction, resulting in the Hamiltonian

$$H = \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{2j}} \left(a^\dagger + a\right) \left(J_+ + J_-\right), \qquad (1)$$

where a and $a^\dagger$ are bosonic annihilation and creation operators, λ is the strength of the Qdot-field coupling and we have set $\hbar = 1$. This Hamiltonian is well–known in quantum optics as the Dicke Hamiltonian, where it serves as a model of an atomic collection interacting with light [6, 8].

2.1. Phase transition

Associated with Hamiltonian (1) is a conserved parity Π such that $[H, \Pi] = 0$, which is given by

$$\Pi = \exp\left\{i\pi\left[a^\dagger a + J_z + j\right]\right\}. \qquad (2)$$

and has eigenvalues of ± 1. In the thermodynamic limit, $N, j \to \infty$, the system undergoes a mean–field type superradiance phase transition [9, 10]. This phase transition persists at zero temperature where it is seen to occur at a critical value of the coupling $\lambda_c = \sqrt{\omega\omega_0}/2$, where the Π symmetry becomes spontaneously broken and the model becomes exactly soluble [7]. Some key properties of the ground state above and below this QPT are listed in Table 1. Below λ_c, the field and the Qdots are unexcited but above λ_c, they both obtain macroscopic excitations, with the resultant ground state being a highly collective state, similar to a polariton mode. Although in what follows we shall mainly be interested in the system at finite N, we shall see that the existence of this phase transition in the thermodynamic limit is crucial to the understanding the appearance of quantum chaos even for relatively small N. This is because the precursors of the QPT give rise to a qualitative change in the nature of the wavefunctions of the system.

Unless otherwise stated, we shall always work on scaled resonance ($\omega = \omega_0 = 1$), which means that the QPT occurs at $\lambda_c = 0.5$.

3. The onset of quantum chaos

To investigate the appearance of quantum chaos, we shall consider the distribution $P(S)$ of the nearest-neighbour level-spacings, $S_n = E_{n+1} - E_n$. That the analysis of such a measure should provide an indicator of quantum chaos is due to the following argument [11]. Classically integrable systems have high degrees of symmetry and hence their quantum counterparts have many conserved quantum numbers. This permits level–crossings to occur in the spectrum, leading to a $P(S)$ with a maximum at small level–spacing, $S \to 0$. The appropriate distribution is Poissonian $P_{\mathrm{P}}(S) = \exp(-S)$, and we shall call quantum spectra with Poissonian statistics "quasi-integrable". On the other hand, classically chaotic systems

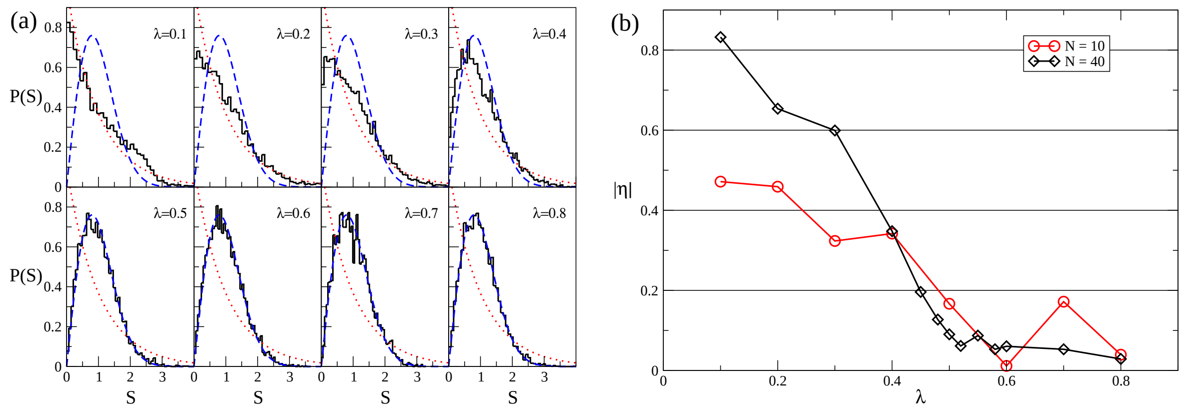

Figure 1. **(a)** Nearest-neighbour distribution $P(S)$ for $N = 40$ at a range of couplings $0.1 \leq \lambda \leq 0.8$. Also plotted are the Poisson (dots) and Wigner (dashes) distributions. **(b)** The modulus of η, Eq. (3), plotted as a function of coupling for systems of $N = 10$ and $N = 40$ Qdots. A value of $\eta = 1$ indicates Poissonian statistics and $\eta = 0$ corresponds to Wigner–Dyson. The system is on scaled resonance ($\omega = \omega_0 = 1$).

have no such integrals of motion and we thus expect their quantum energy spectra to be absent of crossings, leading to $P(S) \to 0$ as $S \to 0$. Although the precise form of the $P(S)$ depends on the symmetries of the model, we shall only have cause to consider the Wigner distribution $P_{\mathrm{W}}(S) = \pi S/2 \exp\left(-\pi S^2/4\right)$ here [12]. To investigate the level statistics of the system, we numerically diagonalise the Hamiltonian (1) in the basis $\{|n\rangle \otimes |j,m\rangle\}$, where $a^\dagger a\,|n\rangle = n\,|n\rangle$ and $|j,m\rangle$ are the Dicke states $J_z\,|j,m\rangle = m\,|j,m\rangle$, and restrict ourselves to the positive parity subspace by only considering states with $n+m+j$ even. We unfold the resulting energy spectrum to rid it of secular variation and construct the distribution function $P(S)$. We then normalise the results for comparison with the generic Poissonian and Wigner distributions described above.

Figure 1a demonstrates the behaviour of the $P(S)$ distribution as a function of coupling for large and fixed $N = 40$. At low couplings ($\lambda \lesssim 0.1$) the $P(S)$ closely resembles the Poisson distribution, $P_P(S)$. As λ is increased ($0.1 < \lambda < 0.5$), this similarity diminishes, but as $P(0)$ remains non-zero, a moderate amount of level-crossing still occurs. This situation changes at and above $\lambda = 0.5 = \lambda_c$, where the $P(S)$ changes to being very well described by the Wigner surmise $P_W(S)$.

The nature of the change in the $P(S)$ distribution in such a cross-over from quasi-integrability to chaos may fruitfully be characterised by the quantity

$$\eta \equiv \frac{\int_0^{S_0}\left[P(S) - P_W(S)\right] dS}{\int_0^{S_0}\left[P_P(S) - P_W(S)\right] dS}, \tag{3}$$

where $S_0 = 0.472913\ldots$ is the value of S at which the two generic distributions $P_P(S)$ and $P_W(S)$ first intersect [13]. η measures the degree of similarity of the measured $P(S)$ with the Wigner surmise $P_W(S)$, and is normalised such that when $P(S) = P_W(S)$ then $\eta = 0$, and when $P(S) = P_P(S)$ then $\eta = 1$. The behaviour of η as a function of coupling for $N = 40$ is shown in Figure 1b, highlighting that the spectrum is predominantly Poissonian at low couplings and that the distribution becomes mixed until at and above $\lambda \approx \lambda_c$ where the spectrum becomes very close to $P_W(S)$. Note that for $\lambda < \lambda_c$ the value of η changes steadily with coupling, whereas above λ_c it becomes relatively constant. Also plotted in this figure is the behaviour of η for a system of $N = 10$ Qdots. In this case, a similar transition is observed but it is not as pronounced as in the $N = 40$ case.

The $P(S)$ distribution is plotted in Figure 2a for a selection of different values of N and

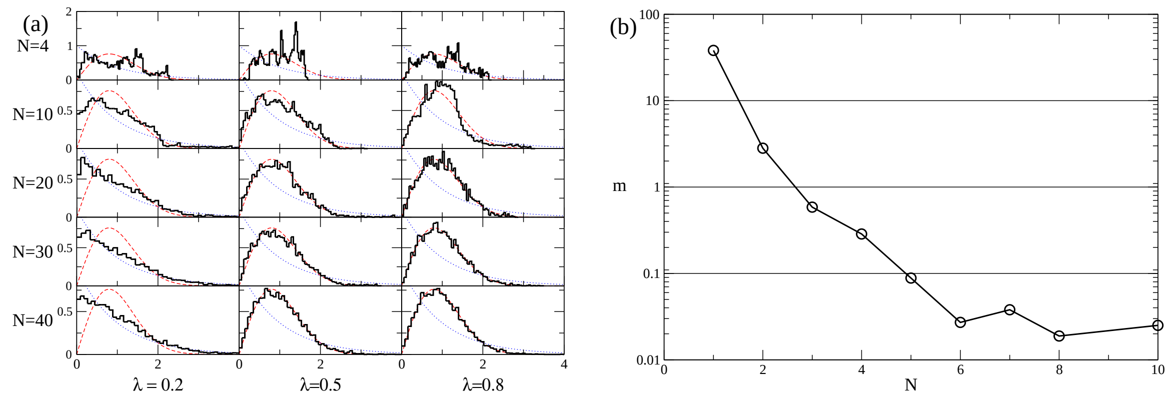

Figure 2. **(a)** Nearest-neighbour distribution $P(S)$ for various values of N and λ, compared with the Poisson (dots) and Wigner (dashes) distributions. **(b)** The mean-square deviation, m, of $P(S)$ from $P_W(S)$, Eq. (4), as a function of the number of Qdots N with a fixed value of coupling $\lambda = \lambda_c$. Note the logarithmic scale. The system is on scaled resonance ($\omega = \omega_0 = 1$).

λ. This figure demonstrates that in both of the phases of the system, increasing N increases the agreement between the calculated $P(S)$ and the respective universal distribution. It is important to note that for very low numbers of co-operating Qdots N, $P(S)$ does not correspond to any of the generic distributions irrespective of coupling, but rather to non-generic forms, characteristic of one-dimensional or harmonic oscillator systems [14].

Finally, in Figure 2b we show how the statistics converge on to the Wigner distribution as a function of N at the fixed value of $\lambda = \lambda_c$, which is the smallest value possible of yielding chaos. Here we do not plot η, as the significance of this measure for a $P(S)$ far removed from the universals is negligible. Rather, we plot the mean-square deviation of the calculated $P(S)$ from $P_W(S)$,

$$m \equiv \int_0^\infty [P_W(S) - P(S)]^2 \, dS. \tag{4}$$

This clearly demonstrates the large deviation of the $P(S)$ from the generic distributions for very low N, and shows a rapid decreases as N is increased. It is apparent that we need to have $N \geq 6$ co-operating Qdots to be in a regime where the observation of quantum chaos would be possible.

4. Discussion, possible experimental realisation

We have seen that for a large enough number of Qdots, the level statistics of the system exhibit a change from Poisson to Wigner statistics at approximately $\lambda = \lambda_c$, corresponding to the rapid onset of quantum chaos as the coupling is increased. The correlation between the onset of quantum chaos and the precursors of the phase transition may be understood by considering the wavefunctions of the system [7]. When viewed in an abstract position representation, one sees that the wavefunctions change at λ_c from being well localised to delocalised, and that the degree of delocalisation is proportional to $\sqrt{N}$, provided that N is sufficiently large. This delocalisation is manifested in the ground state by the appearance of a Schrödinger cat, composed of two parts macroscopically separated from one another. Localisation-delocalisation transitions are consistent with the appearance of chaos [15] as states that are localised on a certain length scale will generally have no overlap with other localised states outside this length, and hence have no reason to exhibit level repulsion. For delocalised states, this is no longer true and level repulsion becomes a possibility. For the smallest values of N, this delocalisation is not possible, and the onset of chaos is not observed. This points to the possibility of using the level-statistics as a method to distinguish whether the Qdots in a many Qdot system are acting collectively with the boson field or not.

One of our original motivations was to understand how precursors of a QPT in finite systems of interacting particles influence the cross-over between quantum chaos and quasi-integrable dynamics. Quantum optical models are a natural choice to study this problem. It has turned out recently, however, that such models are also of high interest in semiconductor physics, one prominent example being the use of the Dicke model in the quest for exciton polariton condensation [16]. On the semiconductor optics side, semiconductor cavity QED [17] can be regarded as a well-established field of physics by now.

Another and perhaps less known possible realisation of the Dicke model are arrays of double quantum dots embedded into a phonon cavity, i.e. a freestanding nanostructure where phonon modes split into subbands and lead to highly non-linear effects in their coupling to electrons [18]. The importance of electron-phonon coupling for transport spectroscopy in double quantum dots is well-known [2, 19, 20], and phonon cavities have already been successfully realized experimentally [21, 22, 23]. A peculiar consequence of the boundary conditions for vibration modes in phonon cavities is the existence of van-Hove singularities in the phonon density of states, i.e., an enormous enhancement of the effective electron-phonon coupling as compared to the bulk case [18].

Acknowledgments

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