Entanglement and the Phase Transition in Single-Mode Superradiance

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We consider the entanglement properties of the quantum phase transition in the single-mode superradiance model, involving the interaction of a boson mode and an ensemble of atoms. For an infinite size system, the atom-field entanglement diverges logarithmically with the correlation length exponent. Using a continuous variable representation, we compare this to the divergence of the entropy in conformal field theories and derive an exact expression for the scaled concurrence and the cusplike nonanalyticity of the momentum squeezing.

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Entanglement has shot to prominence in recent years on the back of the success of three key areas: quantum computing, quantum cryptography, and quantum teleportation. In this quantum information paradigm, entanglement is a resource which can be exploited to perform hitherto unimagined physical tasks.

Latterly, a new emphasis has emerged in which entanglement is related to properties of interacting many-body systems. This approach is being pursued most vigorously in connection with quantum phase transitions (QPTs) [1], as it is hoped that entanglement may shed light upon the dramatic effects occurring in critical systems which, by their very nature, involve complex collective quantum mechanical behavior. A complete theory of many-body entanglement is still lacking. Current techniques are reliant upon bipartite decompositions of the total system, and the criteria for selecting the most pertinent decomposition are by no means clear.

Investigations so far have therefore been restricted to interacting spin-$1/2$ systems on a one-dimensional lattice [2–5] or on a simplex [6], which require the (more or less artificial) splitting into two spin subsystems.

In this Letter, we study the entanglement properties of the one-mode superradiance (Dicke) model [7], where collective and coherent behavior of pseudospins (atoms) is induced by coupling (with interaction constant λ) to a *physically distinct* single-boson subsystem. We present here exact solutions for the entanglement between these two subsystems, and for the pairwise entanglement between atoms at and away from the critical point λ_c. Recently the QPT in this model has been related to the emergence of chaos for $\lambda > \lambda_c$ in a corresponding classical Hamiltonian [8]. Our real-space representation of the modes allows us to analyze the scaling of the atom-field entanglement *at* the critical point and to compare with results from conformal field theories for one-dimensional spin chains [4]. Furthermore, we derive explicit expressions for the concurrence and the related (momentum) squeezing for all coupling parameters λ.

A model that has drawn considerable interest in the context of entanglement near criticality is the *XY* model. In ferromagnetic spin-$1/2$ chains, the concurrence as a function of system size been used [2] to demonstrate scaling of entanglement near the transition point. Osterloh *et al.* [2] have shown that the derivatives of the concurrence between neighbor and next-nearest neighbor spins exhibits a universal scaling behavior in the region of the critical point in this model. Furthermore, the study of such systems has led Osborne and Nielsen [3] to the notion of a "critically entangled" system where the correlation length ξ of the system is divergent and entanglement exists over all length scales. Vidal *et al.* [4] have used an alternative approach and studied the entanglement between blocks of L contiguous spins and the rest of the chain and have found a striking relation to the entropy $S_L \approx (c + \bar{c})/6 \log_2 L + \text{const}$ in $1 + 1$ conformal field theories with central charges c and $\bar{c}$.

We start by describing our model, which is the single-mode Dicke Hamiltonian describing the interaction of N two-level atoms of splitting ω_0 with a single bosonic mode of frequency ω,

$$\begin{aligned}\mathcal{H} &= \omega_0 \sum_{i=1}^{N} s_z^{(i)} + \omega a^\dagger a + \sum_{i=1}^{N} \frac{\lambda}{\sqrt{N}} (a^\dagger + a)(s_+^{(i)} + s_-^{(i)}) \\ &= \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{2j}} (a^\dagger + a)(J_+ + J_-), \end{aligned} \qquad (1)$$

where the second form follows from the introduction of collective spin operators of length $j = N/2$. In the thermodynamic limit, $N, j \to \infty$, the system undergoes a QPT at a critical coupling of $\lambda = \lambda_c = \sqrt{\omega\omega_0}/2$, at which point the system changes from a largely unexcited normal phase to a superradiant one in which both the field and atomic collection acquire macroscopic occupations.

Similar to the large-spin problem analyzed in this context [6], the Dicke Hamiltonian can be regarded as a zero-dimensional field theory with mean-field-type behavior, where the S_N permutation symmetry of the atoms and the absence of an intrinsic length scale makes the model exactly solvable. Despite this simplicity, our model exhibits many nontrivial properties; in particular, exact solutions for the nonanalyticities of the entanglement and the concurrence can be related to the scaling exponent, the finite-size behavior, and the underlying semiclassical integrable/chaos crossover which has been shown to occur around the phase transition [8].

The starting point for our analysis in the thermodynamic limit is the Holstein-Primakoff representation [9] of the angular momentum operators $J_z = (b^\dagger b - j)$, $J_+ = b^\dagger\sqrt{2j - b^\dagger b}$, $J_- = J_+^\dagger$. Here, b and $b^\dagger$ are bosonic operators that convert $\mathcal{H}$ into a two-mode bosonic problem. This allows us to obtain effective Hamiltonians that are exact in the thermodynamic limit, by neglecting terms from expansions of the Holstein-Primakoff square roots [8]. In the normal phase, $\lambda < \lambda_c$, we expand the square roots directly and obtain the effective Hamiltonian

$$\mathcal{H}^{(1)} = \omega_0 b^\dagger b + \omega a^\dagger a + \lambda(a^\dagger + a)(b^\dagger + b) - j\omega_0. \tag{2}$$

In the superradiant phase, we first displace both boson modes by quantities proportional to $\sqrt{j}$ before we approximate the square roots. This leads to a second effective Hamiltonian, the form of which is also bilinear and similar to Eq. (2).

We now consider the normal phase ground state in some detail; the superradiant phase results following with slight modification. The eigenstates of $\mathcal{H}^{(1)}$ are two-mode squeezed states. Via the introduction of a position-momentum representation for the two oscillators, $x \equiv (1/\sqrt{2\omega})(a^\dagger + a)$, $y \equiv (1/\sqrt{2\omega_0})(b^\dagger + b)$, with the momenta defined canonically, we may write the ground-state wave function as

$$\Psi(x, y) = \left(\frac{\varepsilon_+\varepsilon_-}{\pi^2}\right)^{1/4} e^{-(\varepsilon_-/2)(cx-sy)^2-(\varepsilon_+/2)(sx+cy)^2}, \tag{3}$$

where $\varepsilon_\pm^2 = \frac{1}{2}(\omega^2 + \omega_0^2 \pm \sqrt{(\omega_0^2 - \omega^2)^2 + 16\lambda^2\omega\omega_0})$ are the excitation energies of the system, $s \equiv \sin\gamma$, $c \equiv \cos\gamma$, and the angle $2\gamma = \arctan[4\lambda\sqrt{\omega\omega_0}/(\omega_0^2 - \omega^2)]$ characterizes the squeezing axis. This wave function forms the basis of the current analysis.

Atom-field entanglement.—As a measure of the entanglement between the atoms and the field, we calculate the von Neumann entropy $S \equiv -\mathrm{tr}\hat{\rho}\log_2\hat{\rho}$ of the reduced density matrix (RDM) $\hat{\rho}$ of the field mode. In the normal phase, $\hat{\rho}$ is simply determined by the ground-state wave function, Eq. (3), whereas in the superradiant phase *two* degenerate ground states exist that have wave functions $\Psi_\pm$ similar to Eq. (3), but are displaced from the origin by amounts proportional to $\pm\sqrt{j}$. This degeneracy arises from the breaking of the parity symmetry $\Pi = \exp\{i\pi[a^\dagger a + J_z + j]\}$ for $\lambda > \lambda_c$. Because Ψ_+ and Ψ_- are orthogonal, elementary properties of the von Neumann entropy [10] imply that in the superradiant (SR) phase $S(\hat{\rho}_{\mathrm{cat}}) = S(\hat{\rho}_\pm) + 1$, where $\hat{\rho}_\pm$ is the RDM of either of the two (macroscopically separated for large N) solutions, and $\hat{\rho}_{\mathrm{cat}}$ is the RDM of the superposition "cat" state of the two. The cat state restores the broken parity, and thus the latter expression is used for comparison with the numerical results for finite N.

Having clarified this additional distinction between the two phases, we now explicitly calculate the normal phase RDM in the x representation,

$$\rho_L(x, x') = c_L \int_{-\infty}^{\infty} dy f_L(y)\Psi^*(x, y)\Psi(x', y). \tag{4}$$

Here, c_L is a normalization constant, and the introduction of the cutoff function $f_L(y) \equiv e^{-y^2/L^2}$ allows us to discuss the effect of a partial trace over the atomic (y) modes (see below). A straightforward calculation shows that ρ_L is identical to the density matrix of a single harmonic oscillator with frequency Ω_L in a canonical ensemble at temperature $T \equiv 1/\beta$, where

$$\cosh\beta\Omega_L = 1 + 2\frac{\varepsilon_-\varepsilon_+ + 4(\varepsilon_- c^2 + \varepsilon_+ s^2)/L^2}{(\varepsilon_- - \varepsilon_+)^2 c^2 s^2}. \tag{5}$$

The entropy S_L obtained from ρ_L is thus given by the expression

$$S_L(\zeta) = [\zeta\coth\zeta - \ln(2\sinh\zeta)]/\ln 2, \qquad \zeta \equiv \beta\Omega_L/2. \tag{6}$$

This strikingly simple result allows some interesting observations. First of all, the entropy S_∞ (cutoff $L = \infty$) undergoes a divergence at the critical point as we approach λ_c from either side. In the region near λ_c, the excitation energy ε_- vanishes as $\varepsilon_- \propto |\lambda - \lambda_c|^{2\nu}$, with the exponent $\nu = 1/4$ describing the divergence of the characteristic length $\xi \equiv \varepsilon_-^{-1/2}$. Using $S_\infty(\zeta) = [1 - \ln(2\zeta) + \zeta^2/6]/\ln 2 + O(\zeta^4)$ and $\zeta = \sqrt{\varepsilon_\infty/2} \times [1 + O(\varepsilon_\infty)]$ with $\varepsilon_\infty \equiv 2\varepsilon_-/(\varepsilon_+ s^2 c^2)$, we find that S_∞ diverges logarithmically as $S_\infty \propto -(1/2)\log_2(2\varepsilon_\infty)$ and hence (omitting constants),

$$S_\infty \propto -\nu\log_2|\lambda - \lambda_c| = \log_2\xi, \qquad \nu = 1/4. \tag{7}$$

Thus, the entanglement between the atoms and field diverges with the same critical exponent as the characteristic length—a clear demonstration of critical entanglement.

As we approach λ_c the parameter $\zeta = \hbar\Omega_\infty/k_B T$ of the fictitious thermal oscillator approaches zero, indicating that a *classical* limit of the field RDM is being approached, interpreted either as the temperature T going to infinity, or the frequency Ω_∞ approaching zero. In terms of the original parameters of the system, the dependence of the entropy is through the ratio of energies $\varepsilon_\infty \propto \varepsilon_-/\varepsilon_+$[11]. Note that the classical counterpart of the Dicke Hamiltonian has a classical (cusp) singularity in the catastrophe theory sense [12].

We next compare the analytical result from Eqs. (5) and (6) for the entropy S_∞ (corresponding to completely tracing out the atomic mode) with the corresponding finite N results obtained from numerical diagonalization. Figure 1 shows these results and illustrates the finite-size scaling. Defining λ^{M} as the position of the entropy maximum and S_{M} as the value of the maximum entropy, we observe $\lambda^{\mathrm{M}} - \lambda_c \propto N^{-0.75\pm 0.1}$, and $S_{\mathrm{M}} \propto (0.14 \pm 0.01)\log_2 N$.

The accuracy of the exponents are limited by the available numerical data. The divergence of the entropy is logarithmic due to the symmetric nature of the spin system. The entropy here saturates at a maximum value of $\log_2(N+1)$, in contrast with general spin sytems which saturate at $\log_2(2^N)$ due to their larger Hilbert spaces. This distinction is expected to be important in determining the complexity of classically simulating a quantum phase transition [5,13,14]. An explicit plot of the entropy scaling is shown as an inset in Fig. 1, while the scaling of the position of the maximum point is shown as an inset in Fig. 2.

We next consider the system at the critical point but keep the "tracing parameter" L finite. This corresponds to a situation where the trace over the (atomic) y coordinate is performed over only a finite Gaussian effective region of size L for the atomic wave function. With $\varepsilon_- = 0$, the relevant dimensionless energy scale is now $\varepsilon_L \equiv 2/(L^2\varepsilon_+ c^2)$, and the entanglement entropy diverges as (again omitting constants)

$$S_L \propto -(1/2)\log_2(2\varepsilon_L) = \log_2 L, \qquad L \to \infty. \quad (8)$$

This result can now be compared with a recent calculation by Vidal *et al.* [4] of the critical entanglement of blocks of L spins in one-dimensional interacting XY and XXZ spin-chain models. There, the prefactor for the $\log L$ dependence of S_L at criticality is given by the central charges of the underlying conformal field theory in 1 + 1 dimensions. Note, however, that a direct comparison would require the tracing out of L atoms from the N-atom Hamiltonian (see below) with $N \to \infty$, L fixed, but the general principle is the same. In this context, the Dicke model corresponds to a zero-dimensional field theory and, for $N \to \infty$, is in fact closely related to Srednicki's simple two-oscillator model in his introductory discussion of entropy and area [11].

Pairwise entanglement and concurrence.—To investigate the entanglement within the atomic ensemble, we proceed to consider the "pairwise" entanglement between two atoms. Since these two atoms will be in a mixed state we calculate the concurrence [15]. The absence of an intrinsic length scale in our model simplifies our calculations, enabling us to employ the prescription set out for symmetric Dicke states in [16], in which the reduced density matrix ρ_{12} of any two atoms is expressed in terms of the expectation values of the collective operators, $\langle J_z\rangle$, $\langle J_z^2\rangle$, and $\langle J_+^2\rangle$. We then define the *scaled* concurrence as $C_N \equiv NC$, with $C \equiv \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$, where the λ_i are the square roots of the eigenvalues (in descending order) of $\rho_{12}(\sigma_{1y}\otimes\sigma_{2y}) \times \rho_{12}^*(\sigma_{1y}\otimes\sigma_{2y})$. Wang and Sanders have shown that C_N can be expressed in terms of the Kitagawa-Ueda spin squeezing [17] for symmetric multispin states [18].

We show numerical results for the scaled concurrence C_N in Fig. 2, together with the analytic thermodynamic limit result described below. For all λ and N, C_N is less than that of the pure W state $|j, \pm(j-1)\rangle$, which has $C_N = 2$, the maximum pairwise concurrence of any Dicke state [16]. For small coupling λ, we recognize an N-independent behavior of C_N which may be obtained from perturbation theory in λ as

$$C_N(\lambda \to 0) \sim 2\alpha^2/(1+\alpha^2), \qquad \alpha \equiv \lambda/(\omega + \omega_0). \quad (9)$$

As with the entropy, we can perform a finite scaling analysis of the numerical data. Again, two power law expressions are found for λ^{M} and C_N^{M}; $\lambda^{\mathrm{M}} - \lambda_c \propto N^{-0.68\pm 0.1}$ and $C_N^{\mathrm{M}}(\lambda_c) - C_N \propto N^{-0.25\pm 0.01}$. Plots of this behavior are shown as an inset in Fig. 2.

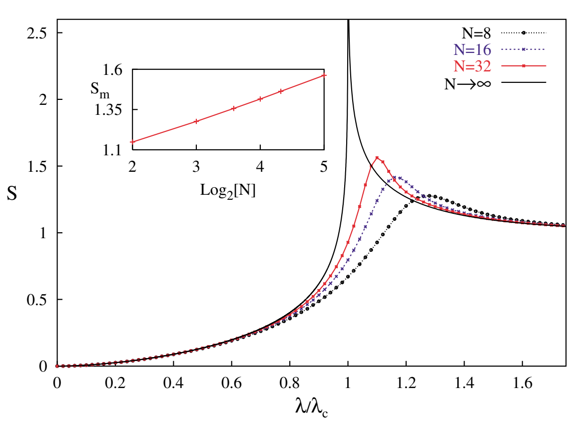

FIG. 1 (color online). Entanglement S_∞ between atoms and field for both $N \to \infty$ and finite N. Inset: scaling of the value of the entanglement maximum as a function of $\log_2 N$. The Hamiltonian is on scaled resonance $\omega = \omega_0 = 1$.

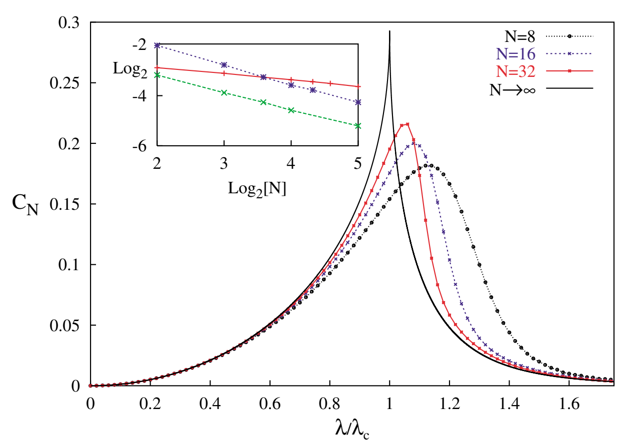

FIG. 2 (color online). Scaled pairwise concurrence $C_N = NC$ between two spins for both $N \to \infty$ and finite N. Inset: scaling of the value (+) and position (×) of the concurrence maximum, and the position of the entropy maximum (*) as functions of N. The Hamiltonian is on scaled resonance $\omega = \omega_0 = 1$.

In the thermodynamic limit $N \to \infty$, the scaled concurrence can be expressed as

$$C_\infty = (1+\mu)[\langle (d^\dagger)^2\rangle - \langle d^\dagger d\rangle] + \tfrac{1}{2}(1-\mu), \qquad (10)$$

setting $\mu = 1$ and $d^\dagger = b^\dagger$ in the normal phase ($\lambda < \lambda_c$), and $\mu = (\lambda_c/\lambda)^2$ and $d^\dagger = b^\dagger + \sqrt{N(1-\mu)/2}$ in the SR phase ($\lambda > \lambda_c$). Recalling $b^\dagger = \sqrt{\omega_0/2}(y - ip_y/\omega_0)$, we can further transform Eq. (9) to establish a relation between the scaled concurrence and the *momentum squeezing* $(\Delta p_y)^2 \equiv \langle p_y^2\rangle - \langle p_y\rangle^2$. We have

$$C_\infty = (1+\mu)[\tfrac{1}{2} - (\Delta p_y)^2/\omega_0] + \tfrac{1}{2}(1-\mu), \qquad (11)$$

where again, setting $\mu = (\lambda_c/\lambda)^2$ gives the superradiant phase equivalent. By using the density matrix of a harmonic oscillator at temperature T, we can calculate an explicit expression for the expectation values in Eq. (10), giving $C_\infty = 1 - (\mu\Omega/\omega_0)\coth(\beta\Omega/2)$. The comparison between the thermal oscillator and the y (atom) density matrix gives $\cosh\beta\Omega = 1 + 2\varepsilon_-\varepsilon_+/D$, $D \equiv [cs(\varepsilon_- - \varepsilon_+)]^2$, and $2\Omega/\sinh\beta\Omega = D/(\varepsilon_- c^2 + \varepsilon_+ s^2)$. Because of symmetry, these are the same parameters as for the reduced field (x) density matrix ρ_∞, Eq. (4), with $s = \sin\gamma$ and $c = \cos\gamma$ interchanged. Using $\coth x/2 = (\cosh x + 1)/\sinh x$ and $s^4 + c^4 = 1 - 2c^2s^2$, after simple algebra one obtains $C_\infty = 1 - \mu(\varepsilon_- s^2 + \varepsilon_+ c^2)/\omega_0$. Because of space restrictions, we give only analytical results at resonance ($\omega = \omega_0$),

$$C_\infty^{x\le 1} = 1 - \tfrac{1}{2}[\sqrt{1+x} + \sqrt{1-x}], \qquad x \equiv \lambda/\lambda_c, \quad (12)$$

$$C_\infty^{x\ge 1} = 1 - \frac{1}{\sqrt{2}x^2}\Big[(\sin^2\gamma)\sqrt{1 + x^4 - \sqrt{(1-x^4)^2+4}} + (\cos^2\gamma)\sqrt{1 + x^4 + \sqrt{(1-x^4)^2+4}}\Big], \qquad (13)$$

where $2\gamma = \arctan[2/(x^2-1)]$ in the SR phase. These explicit expressions reveal the square-root nonanalyticity of the scaled concurrence near the critical point λ_c. The concurrence assumes its maximum $C_\infty = 1 - \sqrt{2}/2 \approx 0.293$ at the critical point $\lambda = \lambda_c$. We note that Eq. (12) is consistent with the maximum of the (unscaled) concurrence approaching the critical point in a related, dissipative version of the Dicke model in the normal phase [19]. Our findings are also in agreement with the behavior of the concurrence in the collective spin model, $H = -(2\lambda/N)(S_x^2 + \gamma S_y^2) - 2S_z + (\lambda/2)(1+\gamma)$ [6], and differ from 1D spin chains, where the maximum of the C does not coincide with its nonanalyticity at the critical point. We also note here that the squeezing obtains its minimal value at λ_c, which is again in agreement with the above spin model.

In conclusion, we have obtained exact results for the entropy and the concurrence in a model that allows us to quantify entanglement across a quantum phase transition. The clear physical distinction between the subsystems (pseudospin or two-level system and bosonic mode) enables us to see distinctly the logarithmic divergence of the entropy in the thermodynamic limit as a function of the coupling constant. We mention that quantum phase transitions have also been discussed very recently in the context of entanglement generation (e.g., for atoms in optical lattices [20]), and quantum computation schemes [14]. The role of phase transitions in the connection between entanglement and underlying integrable to quantum chaotic transitions remains largely unexplored.

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