Entanglement and the Phase Transition in Single Mode Superradiance

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- Entanglement at Quantum Phase Transitions
- Single Mode Superradiance Model
- Outlook

Coworkers: Neill Lambert (Manchester), Clive Emary (Leiden), Jin-Jun Liang (Manchester)

Entanglement

- Quantum Mechanics: there are states with certain quantum correlations between separated observers. Example: EPR-pair

$$\frac{1}{\sqrt{2}}\left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B\right].$$

- How to quantify entanglement?
  - pure states, mixed states; bi-partite, multi-partite.
  - bound states, scattering states; many-body states.
- Entanglement as a tool
  - to generate and control useful quantum correlations.
  - to analyze quantum phase transitions.
  - and the (semi)-classical limit, (quantum) chaos (?)

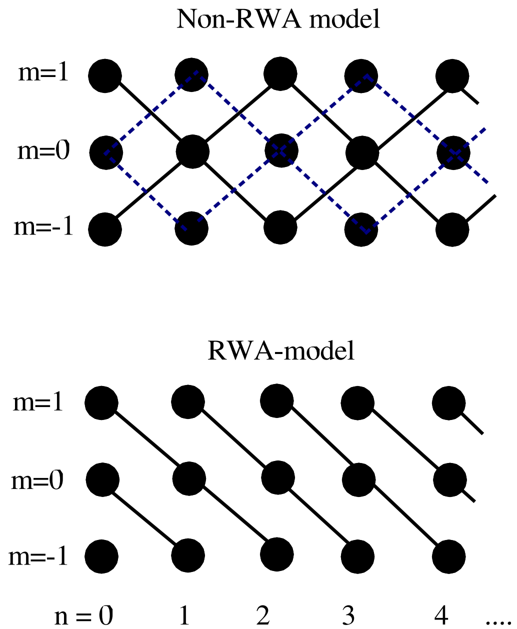
One-mode Superradiance (Dicke) Model

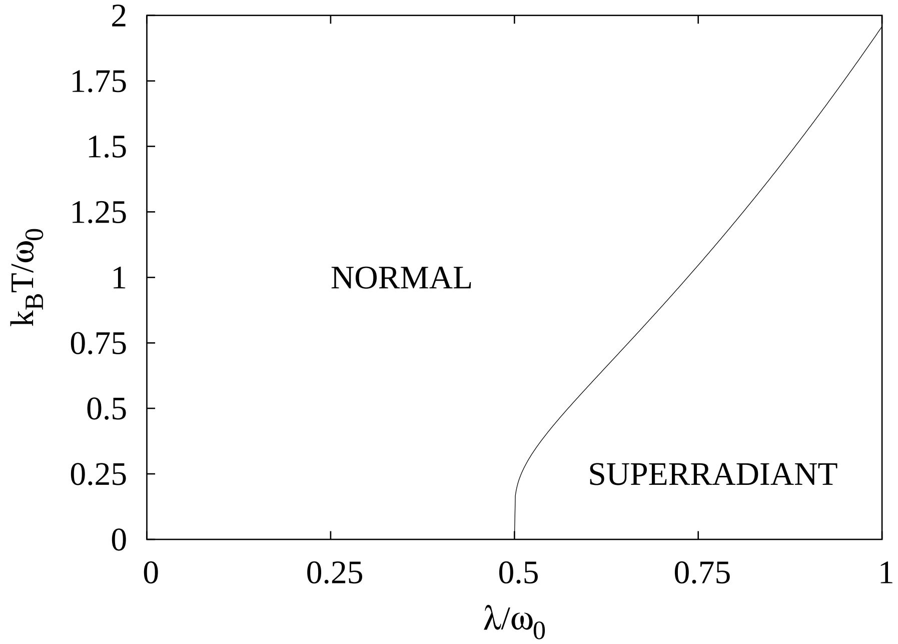
(C. Emary, TB, Phys. Rev. Lett. **90**, 044101 (2003); Phys. Rev. E **67**, 066203 (2003); N. Lambert, C. Emary, TB, Phys. Rev. Lett. **92**, 073602 (2004); quant-ph/0405109.)

- N atoms coupled to single cavity mode (photon, phonon).

$$
\begin{aligned}
\mathcal{H}_{\text{Dicke}} &= \frac{\omega_0}{2}\sum_{i=1}^{N}\sigma_{z,i} + \frac{\lambda}{\sqrt{N}}\sum_{i=1}^{N}\sigma_{x,i}\left(a^\dagger + a\right) + \omega a^\dagger a \\
&= \omega_0 J_z + \frac{\lambda}{\sqrt{2j}}\left(a^\dagger + a\right)\left(J_+ + J_-\right) + \omega a^\dagger a,
\end{aligned}
$$

- Collective spin operators of length $j = N/2$, Dicke states $|jm\rangle$.
- For $N \to \infty$ mean-field type phase transition from normal to superradiant with order parameter $\langle J_z \rangle$ or $\langle a^\dagger a \rangle$; K. Hepp and E. Lieb, Ann. Phys. **76**, 360 (1973); Y. K. Wang and F. T. Hioe, Phys. Rev. A **7**, 831 (1973).





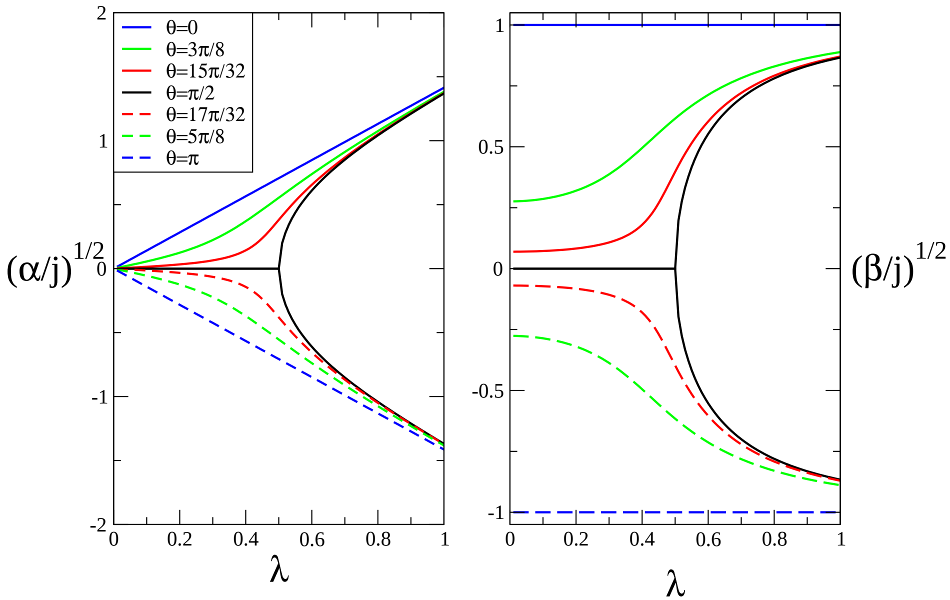
- $T = 0$ quantum phase transition from normal to superradiant at $\lambda = \lambda_c = \sqrt{\omega\omega_0}/2$. Recent CPB-single photon cavity experiments A. Wallraff *et al.*, nature (2004): $N = 1$ and $\lambda \ll \lambda_c$.
- Zero-d field theory; S_N symmetry, no intrinsic length scale: exactly solvable for $N \to \infty$. Non-integrable **chaotic for** $N < \infty$.

Ground State Wave Function

- Holstein-Primakoff representation $J_z = \left(b^\dagger b - j\right)$, $J_+ = b^\dagger \sqrt{2j - b^\dagger b}$.
- Normal phase $\lambda < \lambda_c$: expand square-roots, effective Hamiltonian

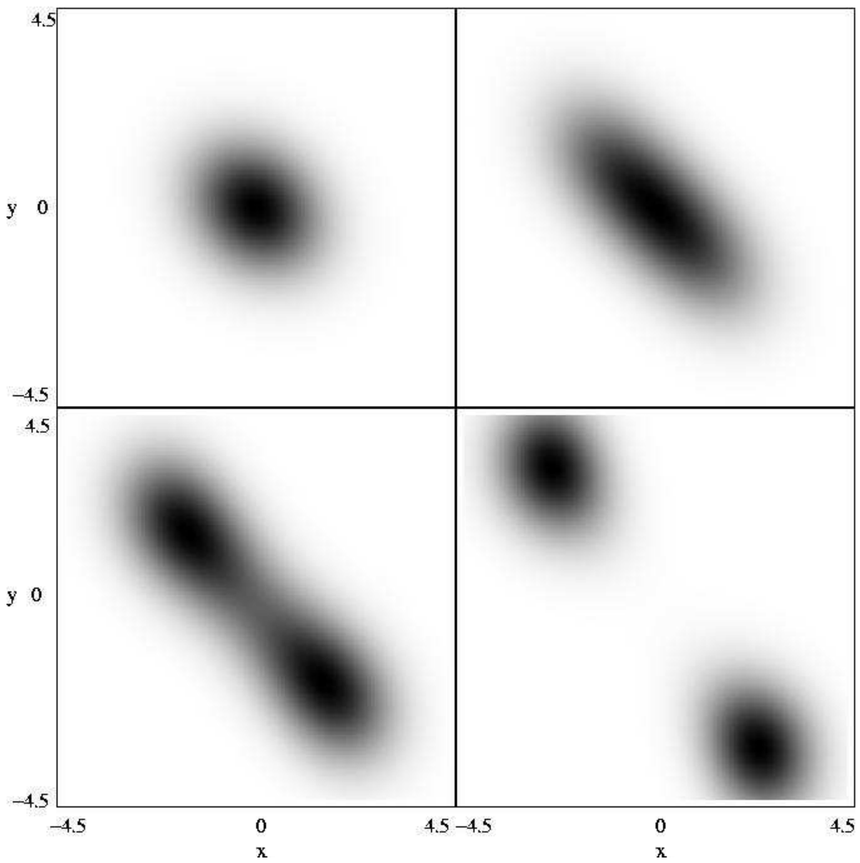
$$\mathcal{H}^{(1)} = \omega_0 b^\dagger b + \omega a^\dagger a + \lambda \left(a^\dagger + a\right)\left(b^\dagger + b\right) - j\omega_0, \quad j \to \infty.$$

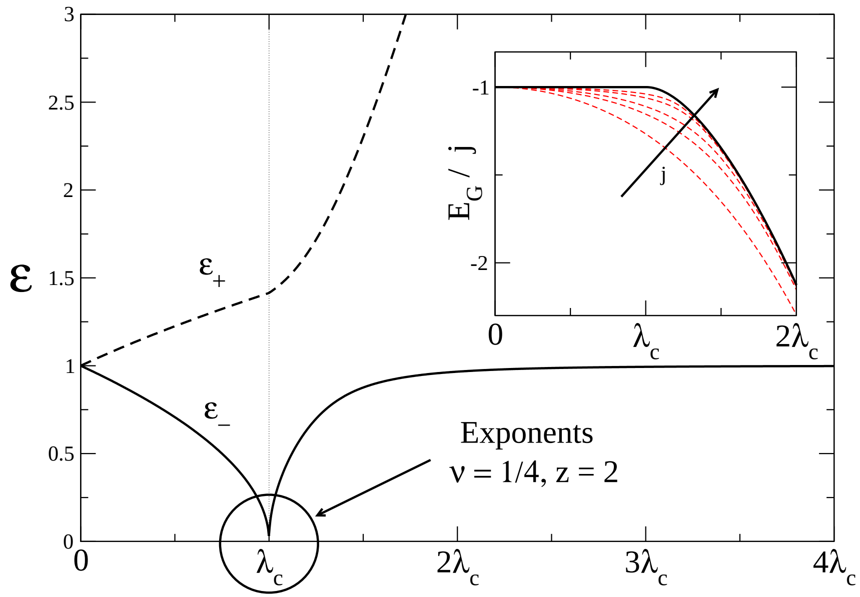
- Super-radiant phase $\lambda > \lambda_c$: boson displacement with $\sqrt{\alpha}, \sqrt{\beta} \propto j$, two equivalent effective Hamiltonians (broken parity symmetry).



(C. Emary, TB; 2004) Order parameters $\alpha = \langle a^\dagger a \rangle$ and $\beta = \langle J_z \rangle + N/2$ for generic large spin-boson Hamiltonians

$$H_\theta = \omega a^\dagger a + \Omega(J_x \cos\theta + J_z \sin\theta) + \frac{2\lambda}{\sqrt{2j}}\left(a^\dagger + a\right) J_x$$



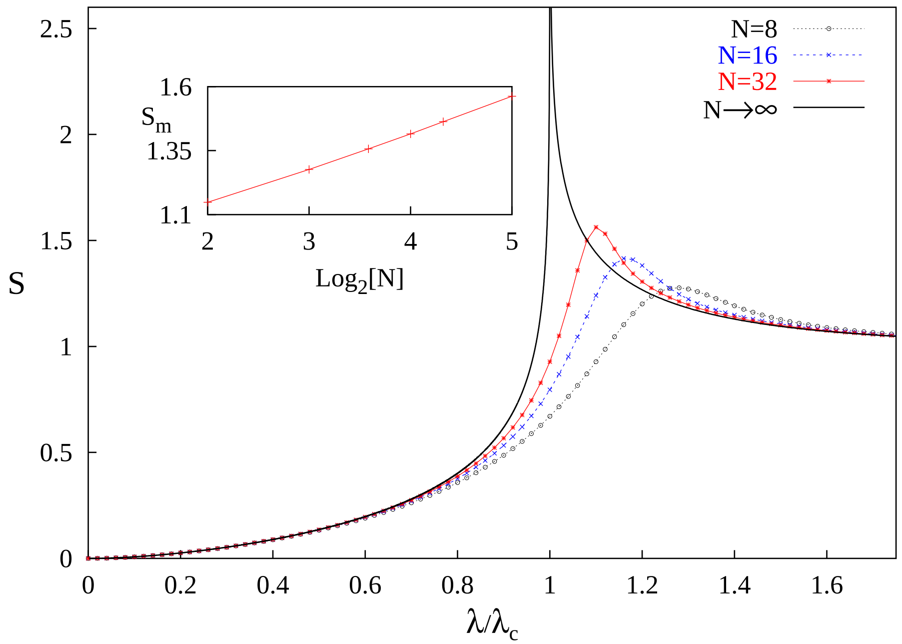


(C. Emary, TB 2003); **Left:** Ground-state $|\psi(x, y)|$ in x-y representation; $x \equiv \frac{1}{\sqrt{2\omega}}\left(a^\dagger + a\right)$ (field mode), $y \equiv \frac{1}{\sqrt{2\omega_0}}\left(b^\dagger + b\right)$ (atom); for $j = 5$ at $\lambda/\lambda_c = 0.2, 0.5, 0.6, 0.7$. **Right:** Excitation energies $\varepsilon_\pm$ for $j \to \infty$. Inset: scaled ground-state energy, E_G/j for $j = 1/2, 1, 3/2, 3, 5, \infty$.

Entanglement between Atoms and Field

- Von-Neumann entropy $S \equiv -\mathrm{tr}\hat{\rho}\log_2\hat{\rho}$ of reduced density matrix (RDM) $\hat{\rho}$ of field-mode.
- RDM in the x-representation,

$$\rho_L(x,x') = c_L \int_{-\infty}^{\infty} dy f_L(y) \Psi^*(x,y)\Psi(x',y), \quad f_L(y) \equiv e^{-y^2/L^2}.$$



ρ_L is density matrix of single harmonic oscillator with frequency Ω_L at temperature $T \equiv 1/\beta$ (N. Lambert, C. Emary, TB, 2004),

$$\begin{aligned}
\cosh\beta\Omega_L &= 1 + 2\frac{\varepsilon_-\varepsilon_+ + 4(\varepsilon_- c^2 + \varepsilon_+ s^2)/L^2}{(\varepsilon_- - \varepsilon_+)^2 c^2 s^2} \\
S_{L=\infty} &= \log_2 \xi + const \\
\xi &\equiv \varepsilon_-^{-1/2} \propto |\lambda - \lambda_c|^{-z\nu/2}, \nu = \frac{1}{4}, z = 2.
\end{aligned}$$

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RESULTS:

- Entropy S_∞ (cut-off $L = \infty$) diverges for $\lambda \to \lambda_c$, $S_\infty \propto -\nu \log_2 |\lambda - \lambda_c| = \log_2 \xi, \quad \nu = 1/4.$
- exponent $\nu = 1/4$ describes divergence of characteristic length $\xi \equiv \varepsilon_-^{-1/2}$.
- For $\lambda \to \lambda_c$, fictitious thermal oscillator parameter $\zeta = \hbar\Omega_\infty / k_B T \to 0$: *classical* limit of the field RDM (temperature $T \to \infty$ or frequency $\Omega_\infty \to 0$.
- Trace over finite y-region of size L only: Entropy $S_L(\zeta) = [\zeta \coth \zeta - \ln(2 \sinh \zeta)] / \ln 2, \quad \zeta \equiv \beta\Omega_L/2.$

$$S_L \propto -(1/2) \log_2(2\varepsilon_L) = \log_2 L, \quad L \to \infty,$$

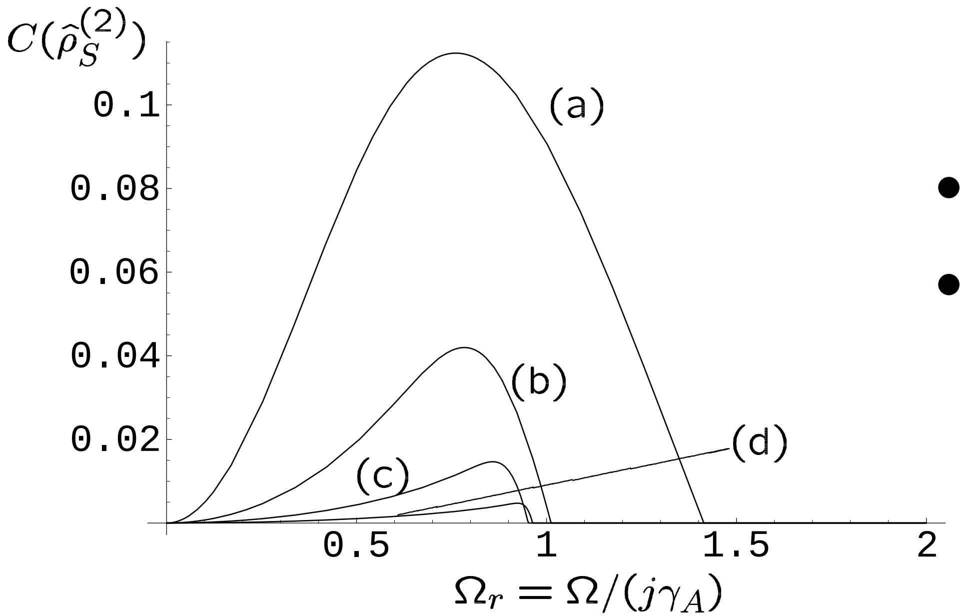
similar to entanglement of blocks of L spins in one-dimensional interacting XY and XXZ spin-chain models: $S_L \approx (c + \bar{c})/6 \log L + const$ in $1+1$ conformal field theories with central charges c and $\bar{c}$. (G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. **90**, 227902 (2003)).

Pairwise Entanglement in Dissipative Dicke Model

R. H. Dicke, Phys. Rev. **93**, 99 (1954)

Schneider and Milburn; Phys. Rev. A **65**, 042107 (2002): driven, dissipative large pseudo-spin model

$$\frac{\partial\rho}{\partial t} = -i\frac{\omega_0}{2}[J_+ + J_-, \rho] + \frac{\gamma_A}{2}\left(2J_-\rho J_+ - J_+J_-\rho - \rho J_+J_-\right)$$



- Steady state.
- Entanglement maxima on the weak coupling side of the transition in *unscaled* two-atom concurrence.

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Pairwise Entanglement in Single-Mode Dicke Model

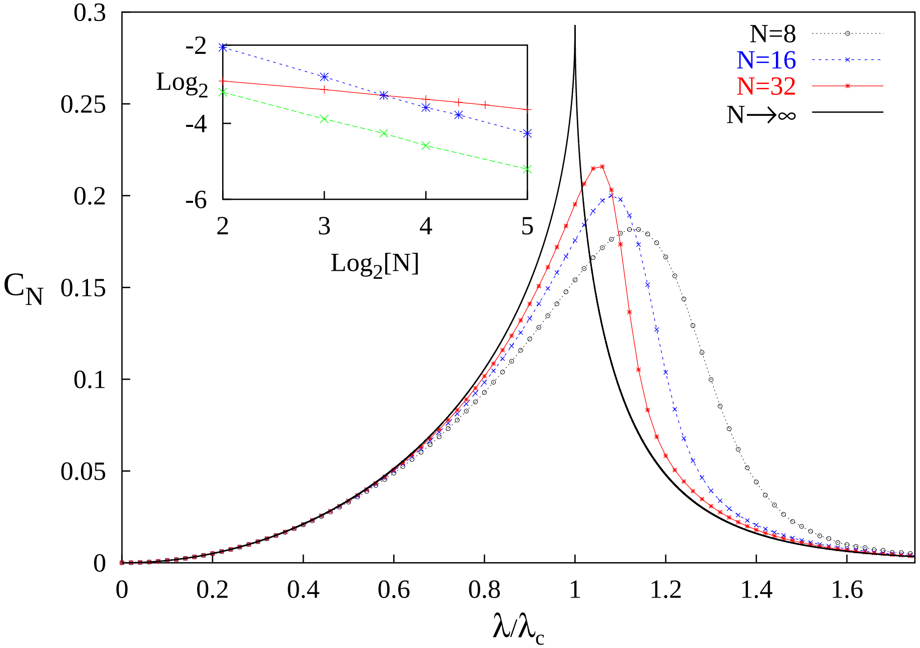
- *Scaled* concurrence $C_N \equiv NC$ as a measure (W. K. Wootters, Phys. Rev. Lett **80**, 2245 (1998)) for pairwise entanglement between 2 atoms (mixed state, boson traced out).

  $C \equiv \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \lambda_j^2 = \text{EV of } \rho_{12}(\sigma_{1y} \otimes \sigma_{2y})\rho_{12}^*(\sigma_{1y} \otimes \sigma_{2y})$

  - S_N symmetry helps (X. Wang and K. Mølmer, Eur. Phys. J. D **18**, 385 (2002).): need reduced density matrix ρ_{12} for any two atoms.
  - Always $C_N < 2 =$ (maximum pairwise concurrence of any Dicke state).

- Perturbation theory: $C_N(\lambda \to 0) \sim 2\alpha^2/(1+\alpha^2), \quad \alpha \equiv \lambda/(\omega + \omega_0)$.

- Relation between scaled concurrence and *momentum squeezing*, $C_\infty = (1+\mu)\left[\frac{1}{2} - (\Delta p_y)^2/\omega_0\right] + \frac{1}{2}(1-\mu)$, $\mu = 1$ in normal phase and $\mu = (\lambda_c/\lambda)^2$ in SR phase. Kitagawa-Ueda (Phys. Rev. A **47**, 5138 (1993)) *spin squeezing* for $\xi^2 \equiv \frac{4}{N}(\Delta \vec{S}\vec{n})^2 < 1$. (X. Wang and B. C. Sanders, Phys. Rev. A **68**, 012101 (2003).).



Concurrence assumes its *maximum* $C_\infty = 1-\sqrt{2}/2 \approx 0.293$ *at* the critical point $\lambda = \lambda_c$ (as in Lipkin model, J. Vidal, G. Palacios, and R. Mosseri; Phys. Rev. A **69**, 022107 (2004); J. Reslen, L. Quiroga, and N. F. Johnson, cond-mat/0406674 (2004)).

$$
\begin{aligned}
C_\infty^{x\le 1} &= 1-\frac{1}{2}\left[\sqrt{1+x}+\sqrt{1-x}\right], \quad x\equiv \lambda/\lambda_c \\
C_\infty^{x\ge 1} &= 1-\frac{1}{\sqrt{2}x^2}\left[\sin^2\gamma\sqrt{1+x^4-\sqrt{(1-x^4)^2+4}}+\cos^2\gamma\sqrt{1+x^4+\sqrt{(1-x^4)^2+4}}\right] \\
2\gamma &= \arctan[2/(x^2-1)] \quad \text{in SR phase.}
\end{aligned}
$$

N. Lambert, C. Emary, TB, Phys. Rev. Lett. **92**, 073602 (2004); quant-ph/0405109, (2004).

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Lipkin-Meshkov-Glick Model Nucl. Phys. **62**, 188 (1965)

- J. Vidal, G. Palacios, and R. Mosseri; Phys. Rev. A **69**, 022107 (2004).

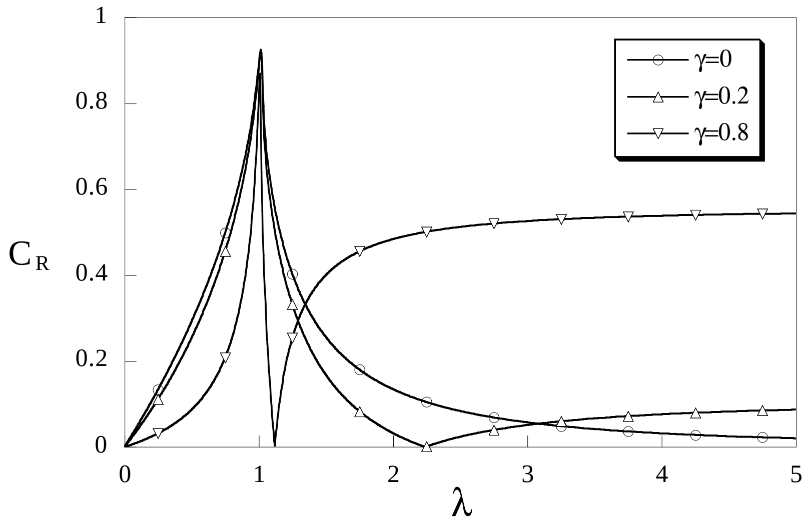
$$H \equiv -\frac{\lambda}{N}\sum_{i<j}^{N}\left(\sigma_x^i\sigma_x^j + \gamma\sigma_y^i\sigma_y^j\right) - \sum_{i=1}^{N}\sigma_z^i$$

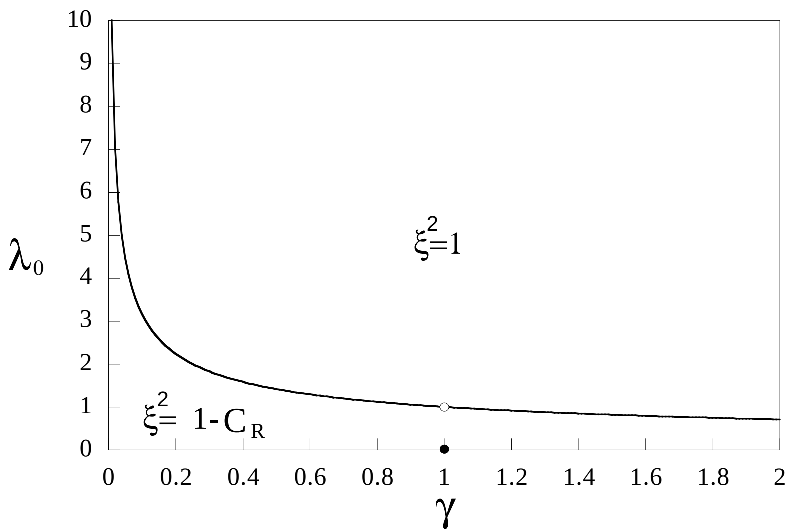
$$= -\frac{2\lambda}{N}\left(J_x^2 + \gamma J_y^2\right) - 2J_z + \frac{\lambda}{2}(1+\gamma), \quad J_\alpha \equiv \frac{1}{2}\sum_{i=1}^{N}\sigma_\alpha^i, \quad \alpha = x, y, z.$$

- 2nd order, mean-field type QPT from nondegenerate to doubly degenerate ground state at $\lambda_c = 1$ for any anisotropy parameter $\gamma \neq 1$.

- Rescaled concurrence $C_N \equiv NC$;

$$1 - C_{N-1}(\lambda_m) \sim N^{-0.33\pm0.01}, \quad \lambda_m - \lambda_c \sim N^{-0.66\pm0.01}, \quad \gamma \neq 1.$$





- Vanishing of the concurrence for $\gamma \neq 0$ at a special value $\lambda_0(\gamma)$. Note $C \equiv \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$. Therefore, non-analyticity in C does not neccessarily indicate QPT (M.-F. Yang, quant-ph/0407226).
- Kitagawa-Ueda spin squeezing for $\xi^2 \equiv \frac{4}{N}(\Delta\vec{S}\vec{n})^2 < 1$.
- For $\gamma = 0$ ground state always spin squeezed although in same universality class as $\gamma \neq 0$.
- Lipkin model is infinitely coupled *XY-model in magnetic field.*

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XY model for spin-$\frac{1}{2}$s on ferromagnetic chain

- A. Osterloh, L. Amico, G. Falci, R. Fazio; nature **416**, 608 (2002). Independent analysis by T. Osborne, M. A. Nielsen, Phys. Rev. A **66**, 032110 (2002).

$$H = -\lambda(1+\gamma)\sum_{i=1}^{N}\sigma_i^x\sigma_{i+1}^x - \lambda(1-\gamma)\sum_{i=1}^{N}\sigma_i^y\sigma_{i+1}^y - \sum_{i=1}^{N}\sigma_i^z,$$

- Concurrence $C(i)$ as entanglement measure between two sites with distance i,

- Correlation length diverges at the critical point, but $C(i>2)$ vanish.

- Non-analyticity of $C(1)$ at $\lambda = \lambda_c$ for $N \to \infty$ in Ising model $\gamma = 1$;

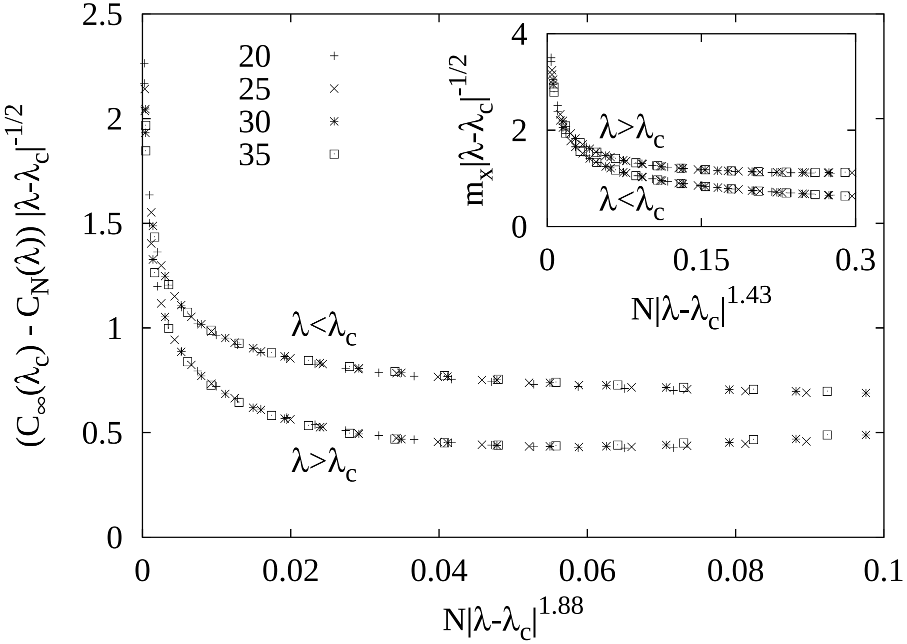
$$dC(1)/d\lambda = (8/3\pi^2)\ln|\lambda - \lambda_c| + \text{const}, \quad N \to \infty,$$

related with precursors at $\lambda = \lambda_m$ for finite N (with $\lambda_m - \lambda_c \propto N^{-1.86}$): single-parameter scaling function $f(N^{1/\nu}(\lambda - \lambda_m))$.

Finite-Size Scaling in Single-Mode Dicke Model

- Position of entropy maximum $\lambda^{\mathrm{M}} - \lambda_c \propto N^{-0.75\pm0.1}$, concurrence maximum $\lambda^{\mathrm{M}} - \lambda_c \propto N^{-0.68\pm0.1}$, $C_N^{\mathrm{M}}(\lambda_c) - C_N \propto N^{-0.25\pm0.01}$.

More detailed analysis by **J. Reslen, L. Quiroga, and N. F. Johnson, cond-mat/0406674**



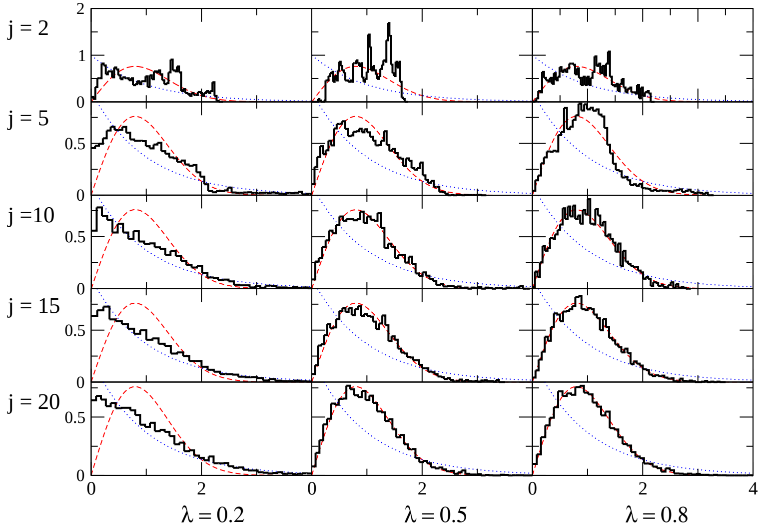
One-parameter scaling analysis

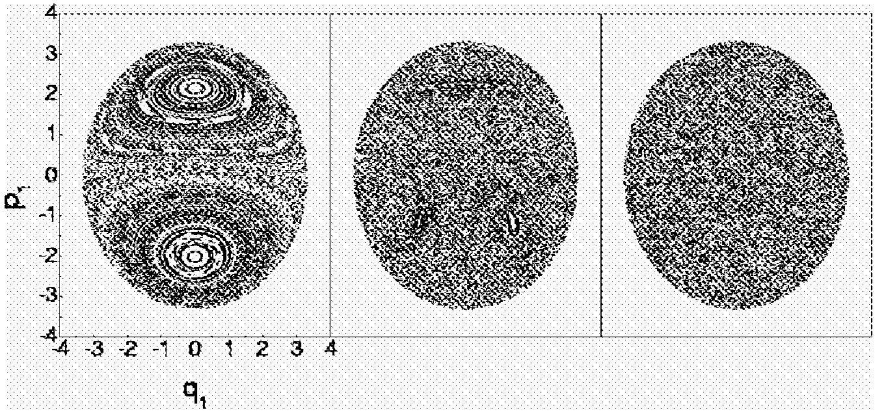
$$C_\infty(\lambda_c) - C_N(\lambda) = |\lambda-\lambda_c|^a f(N|\lambda-\lambda_c|^b)$$

$f(N) \sim N^b$	*Dicke*	*Lipkin*
$C_\infty(\lambda_c) - C_N(\lambda_c)$	-0.26 ± 0.01	-0.30 ± 0.01
$C_\infty^M - C_N^M$	-0.28 ± 0.03	-0.30 ± 0.03
$\lambda_N^M - \lambda_c$	-0.65 ± 0.03	-0.66 ± 0.03

$\rightsquigarrow$ same universality class.

QPT, QChaos, Entanglement





- Level spacing distributions $P(S)$ for $\mathcal{H}_{\rm Dicke}$ at finite $N = j/2$, C. Emary, TB, Phys. Rev. E **67**, 066203 (2003).
- Poincaré sections for the *classical Dicke Model*, cf. also (X.-W. Hou and B. Hu, Phys. Rev. A **69**, 042110 (2004)).
- How to gain more insight?

Further Insight

entanglement

quantum phase transitions | quantum chaos and classical limit

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$\nwarrow$ Quantum kicked rotor $\mathcal{H} = \alpha J_z^2 + \beta J_y \sum_n \delta(t - n\tau)$: X. Wang, S. Ghose, B. C. Sanders, and B. Hu, PRE **70**, 016217 (2004); H. Fujisaki, T. Miyadera, and A. Tanaka, Phys. Rev. E **67**, 066201 (2003).

$\leftrightarrow$ B. Georgeot and D. L. Shepelyansky, PRL **81**, 5129 (1998); PRE **62**, 3504 (2000); C. Emary, TB, Phys. Rev. Lett. **90**, 044101 (2003).

$\nearrow$ Entanglement and bifurcations: A. P. Hines, C. M. Dawson, R. H. McKenzie, and G. J. Milburn , Phys. Rev. A **70**, 022303 (2004). n-level p-mode Dicke superradiance (J.-J. Liang, 2003): additional symmetries.

Summary, Outlook

- Entropy of formation and scaled concurrence for single-mode superradiance model, analytical results for $N \to \infty$.
- OPEN:
  - Scaling exponents for maxima.
  - Multi-partite entanglement.
  - Relation between entanglement and quantum chaos/ level statistics.
  - How 'classical' is the limit $N \to \infty$? Classical limit of entanglement?
  - More models, in particular $d \geq 2$ needed.