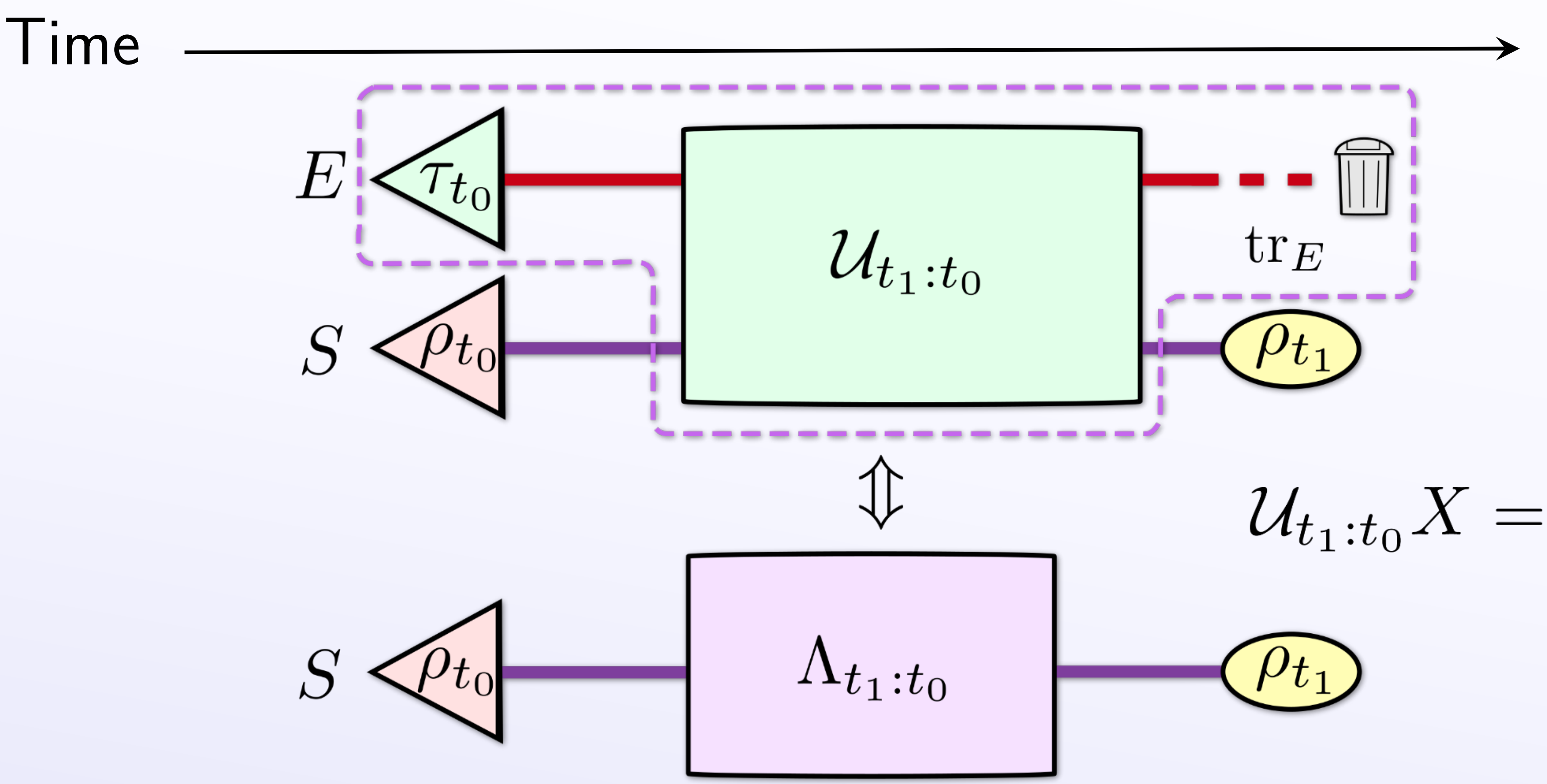
Operational characterisation of non-Markovian quantum processes

Felix A. Pollock

FAP, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro & K. Modi, arXiv:1512.00589
Simon Milz, FAP & K. Modi, arXiv:1610.02152
FAP & K. Modi, *in preparation*

Two pictures of open dynamics



$$\mathcal{U}_{t_1:t_0} X = U_{t_1:t_0} X U^\dagger_{t_1:t_0}$$

$$\rho_{t_1} = \Lambda_{t_1:t_0} \rho_{t_0} = \mathrm{tr}_E\{\mathcal{U}_{t_1:t_0}(\rho_{t_0} \otimes \tau_{t_0})\}$$

What is a non-Markovian process?

'Usual' picture: $\rho_{t_1} = \Lambda_{t_1:t_0}\rho_{t_0} = \mathrm{tr}_E\{\mathcal{U}_{t_1:t_0}(\rho_{t_0} \otimes \tau_{t_0})\}$

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Non-Markovian if e.g.

Indivisible	$\Lambda_{t_2:t_0}\rho_{t_0} \neq \Lambda_{t_2:t_1}\Lambda_{t_1:t_0}\rho_{t_0}$
Distinguishability non-decreasing	$\exists t_2 > t_1 \quad D(\Lambda_{t_2:t_0}\rho_{t_0}, \Lambda_{t_2:t_0}\rho'_{t_0}) \geq D(\Lambda_{t_1:t_0}\rho_{t_0}, \Lambda_{t_1:t_0}\rho'_{t_0})$

$\vdots$

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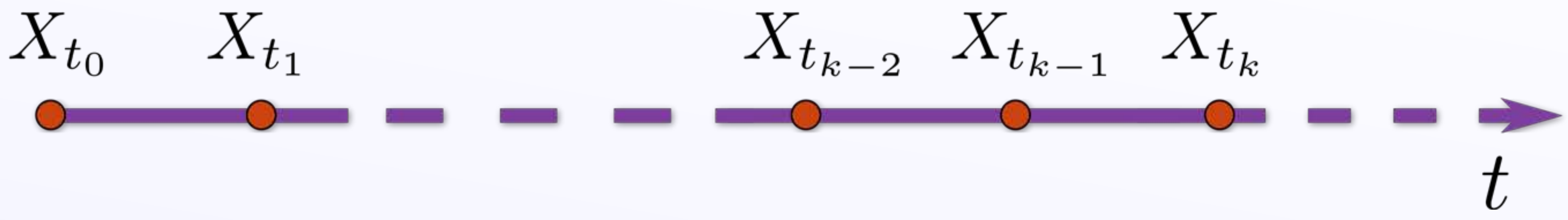
$\vdots$

No comprehensive characterisation of memory effects

What is a non-Markovian process?

Classically: $\{P(X_{t_0}, X_{t_1}, \ldots, X_{t_k}) | k \in \mathbb{N}\}$ + Kolmogorov conditions



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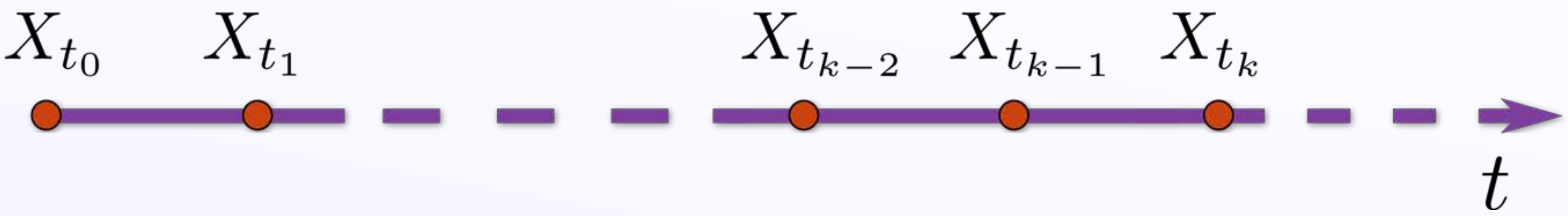
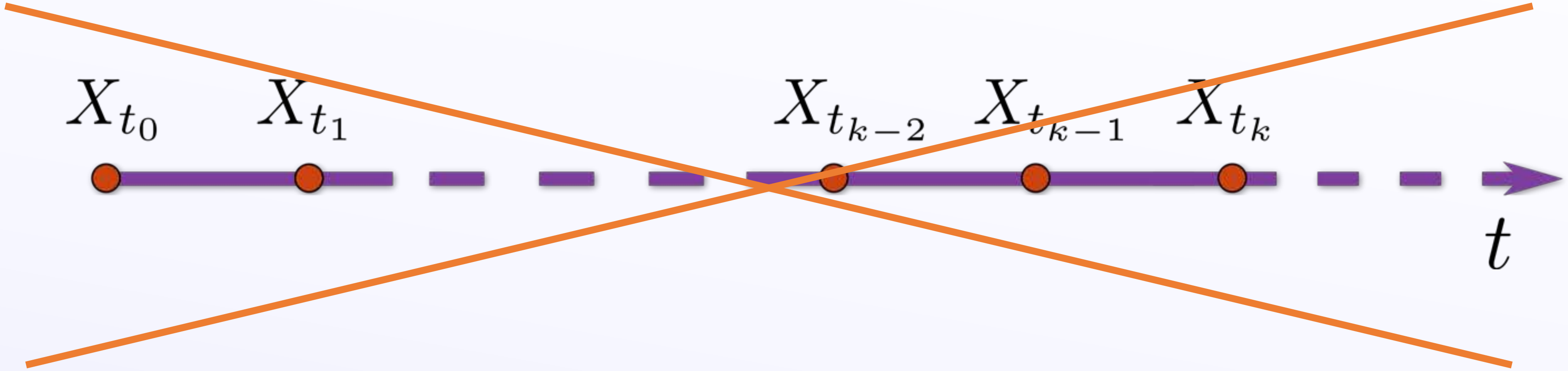


Non-Markovian if

$$P(X_{t_k} | X_{t_0}, X_{t_1}, \ldots, X_{t_{k-1}}) \neq P(X_{t_k} | X_{t_{k-1}})$$

What is a non-Markovian process?

Quantum mechanically:



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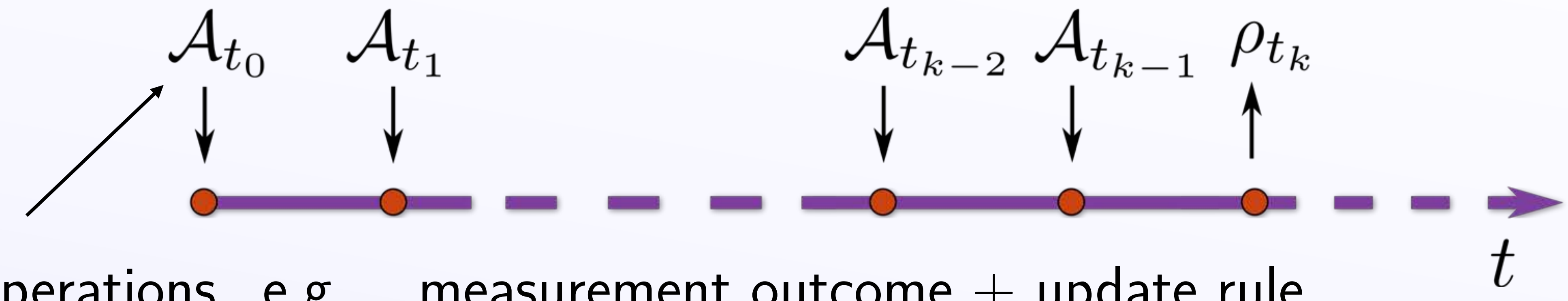

CP operations e.g. measurement outcome + update rule,
unitary transformation, projection etc.

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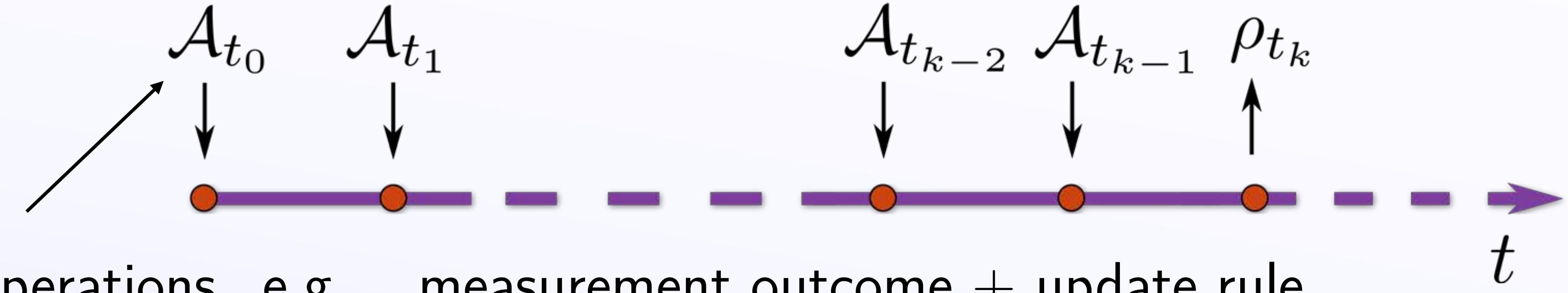

CP operations e.g. measurement outcome + update rule, unitary transformation, projection etc.

$$\rho_{t_k} = \mathcal{T}_{t_k:t_0}[\mathcal{A}_{t_0}, \mathcal{A}_{t_1}, \ldots, \mathcal{A}_{t_{k-1}}]$$

$$\mathrm{tr}\rho_{t_k} = P(\mathcal{A}_{t_0}, \mathcal{A}_{t_1}, \ldots, \mathcal{A}_{t_{k-1}}) \longleftarrow$$ Includes all multitime correlation functions.

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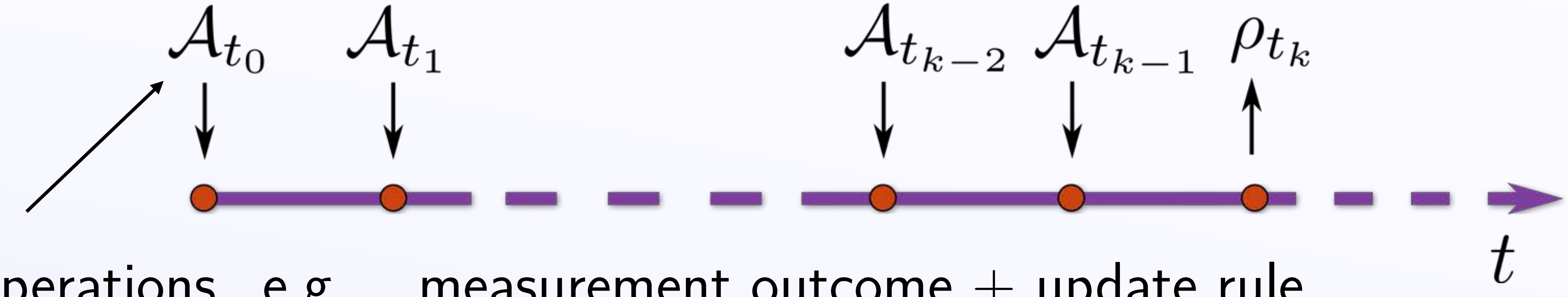

CP operations e.g. measurement outcome + update rule,
unitary transformation, projection etc.

Completely positive linear map.

$$\rho_{t_k} = \mathcal{T}_{t_k:t_0}[\mathcal{A}_{t_0}, \mathcal{A}_{t_1}, \ldots, \mathcal{A}_{t_{k-1}}]$$

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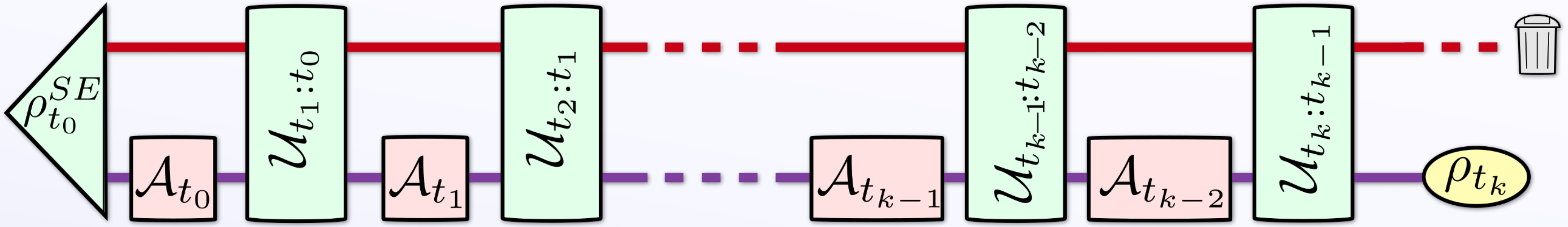
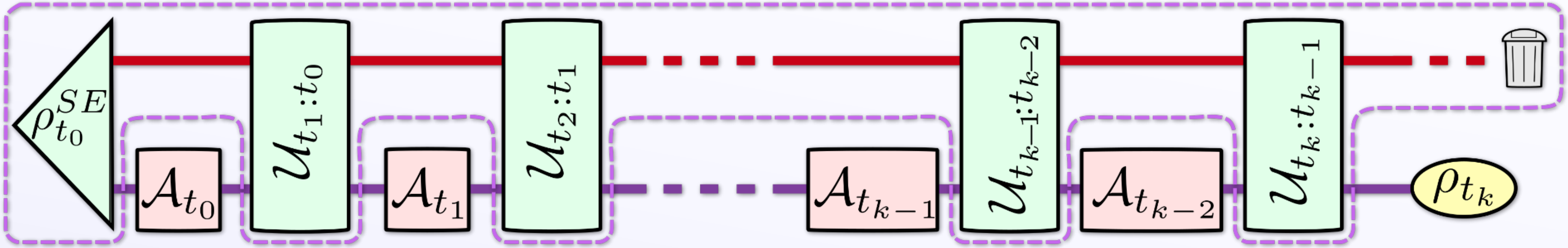


$$\rho_{t_k} = \mathrm{tr}_E \left\{ \mathcal{U}_{t_k:t_{k-1}} (\mathcal{A}_{t_{k-1}} \otimes \mathcal{I}) \cdots (\mathcal{A}_{t_1} \otimes \mathcal{I}) \mathcal{U}_{t_1:t_0} (\mathcal{A}_0 \otimes \mathcal{I}) \rho_{t_0}^{SE} \right\}$$

Process tensor description



$$\begin{aligned}\rho_{t_k} =& \mathcal{T}_{t_k:t_0}[\mathcal{A}_{t_0}, \mathcal{A}_{t_1}, \cdots, \mathcal{A}_{t_{k-1}}] \\ =& \mathrm{tr}_E \left\{ \mathcal{U}_{t_k:t_{k-1}} (\mathcal{A}_{t_{k-1}} \otimes \mathcal{I}) \cdots (\mathcal{A}_{t_1} \otimes \mathcal{I}) \mathcal{U}_{t_1:t_0} (\mathcal{A}_0 \otimes \mathcal{I}) \rho_{t_0}^{SE} \right\}\end{aligned}$$

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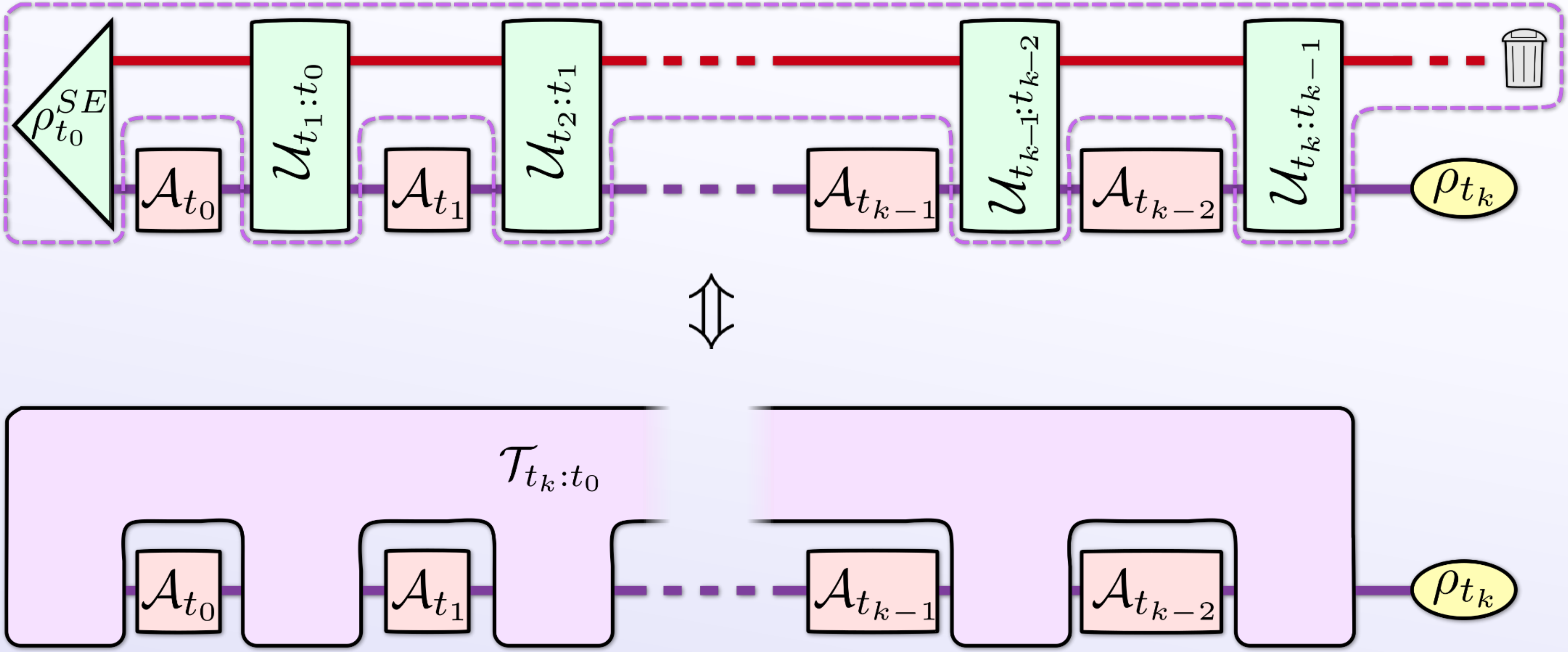

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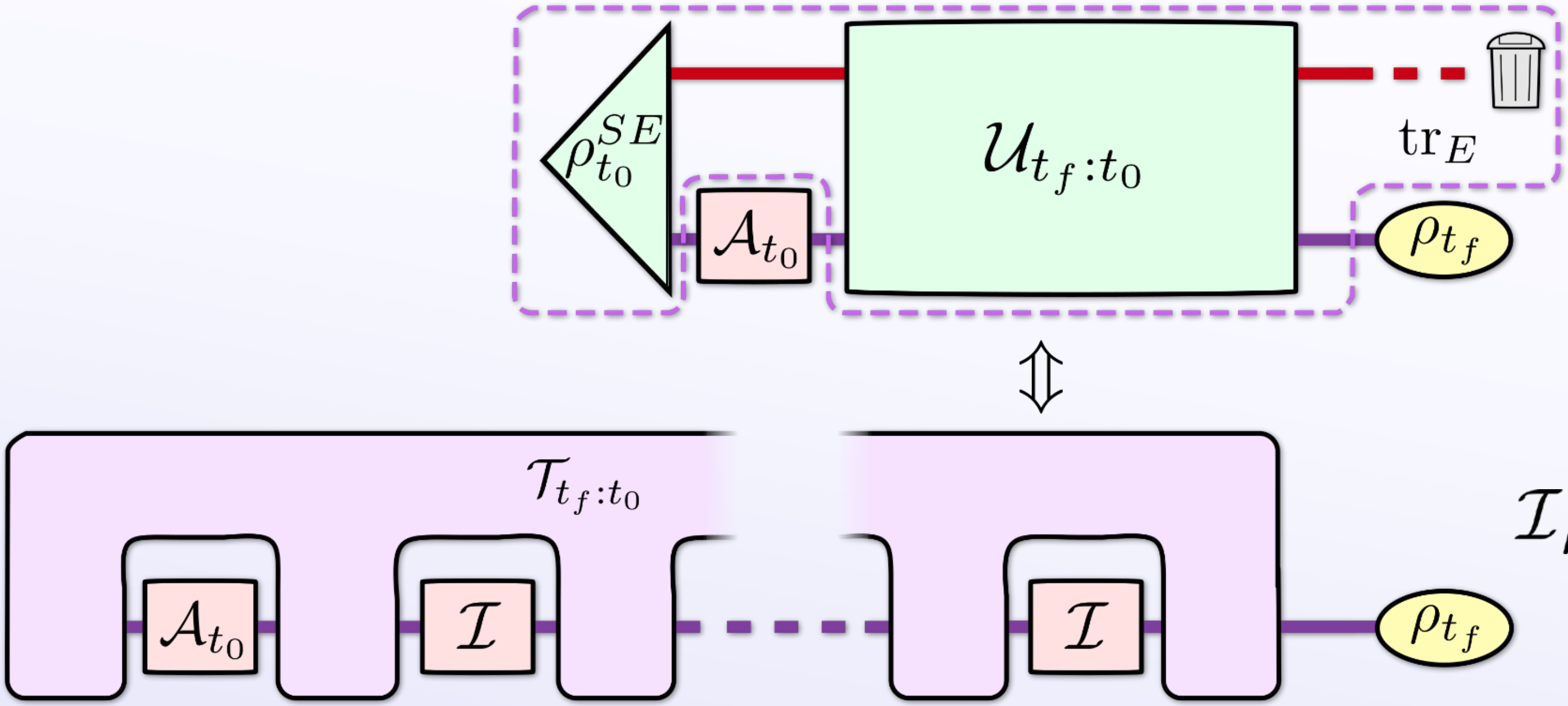
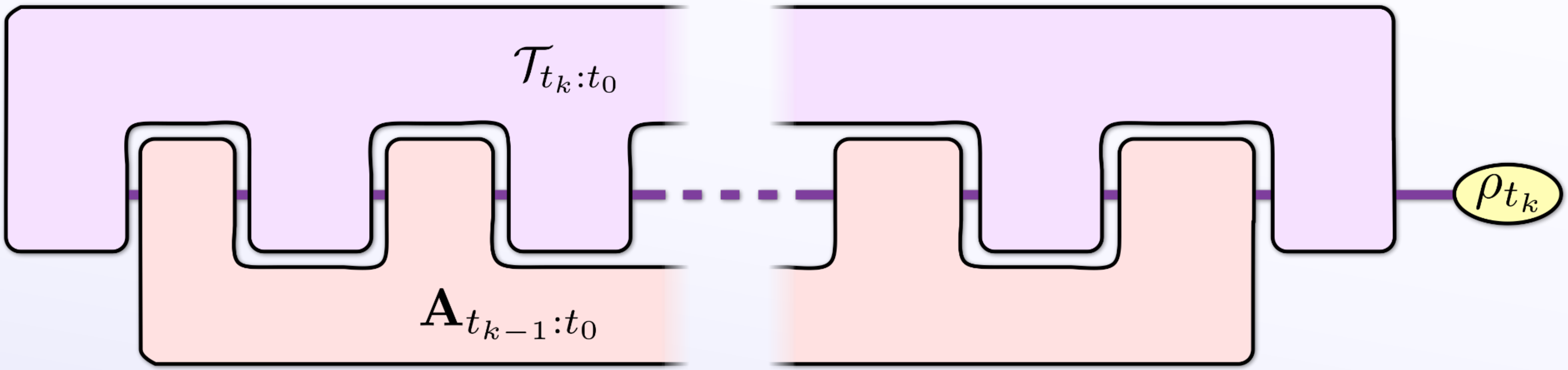


$$\mathcal{I}\rho = \rho$$

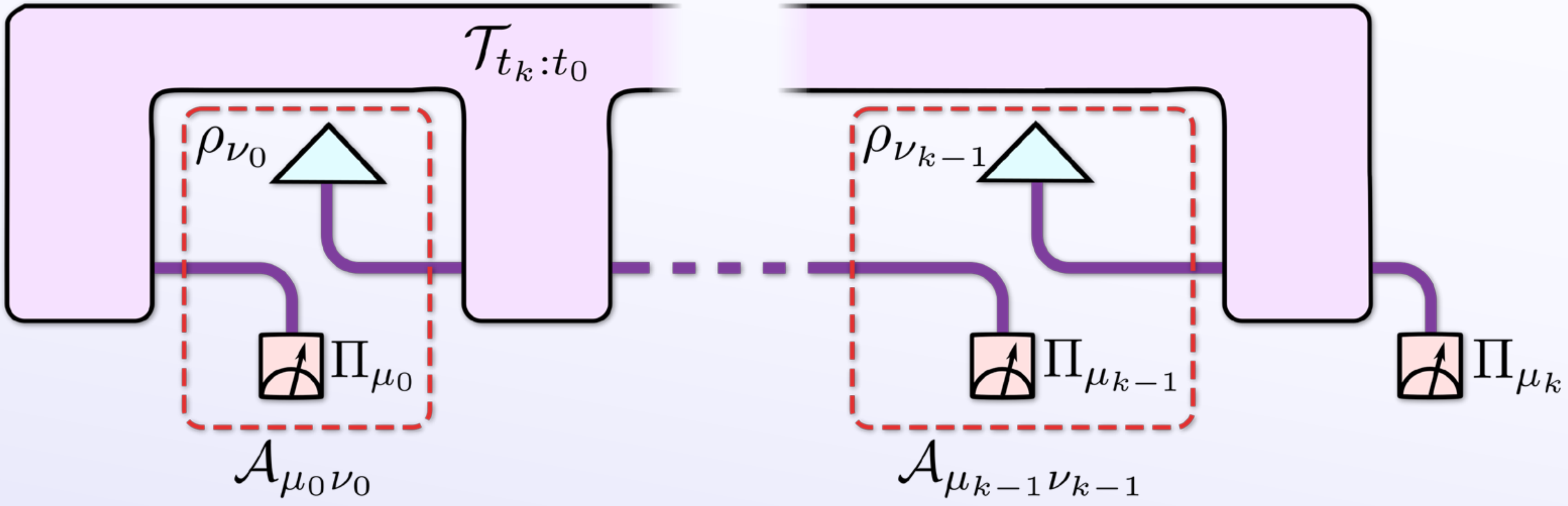
Process tensor description



$$\rho_{t_k} = \mathcal{T}_{t_k:t_0}[\mathbf{A}_{t_{k-1}:t_0}]$$

Process tensor description

$\mathcal{T}_{t_k:t_0}$ can be reconstructed tomographically (scales as d^{4k+2})



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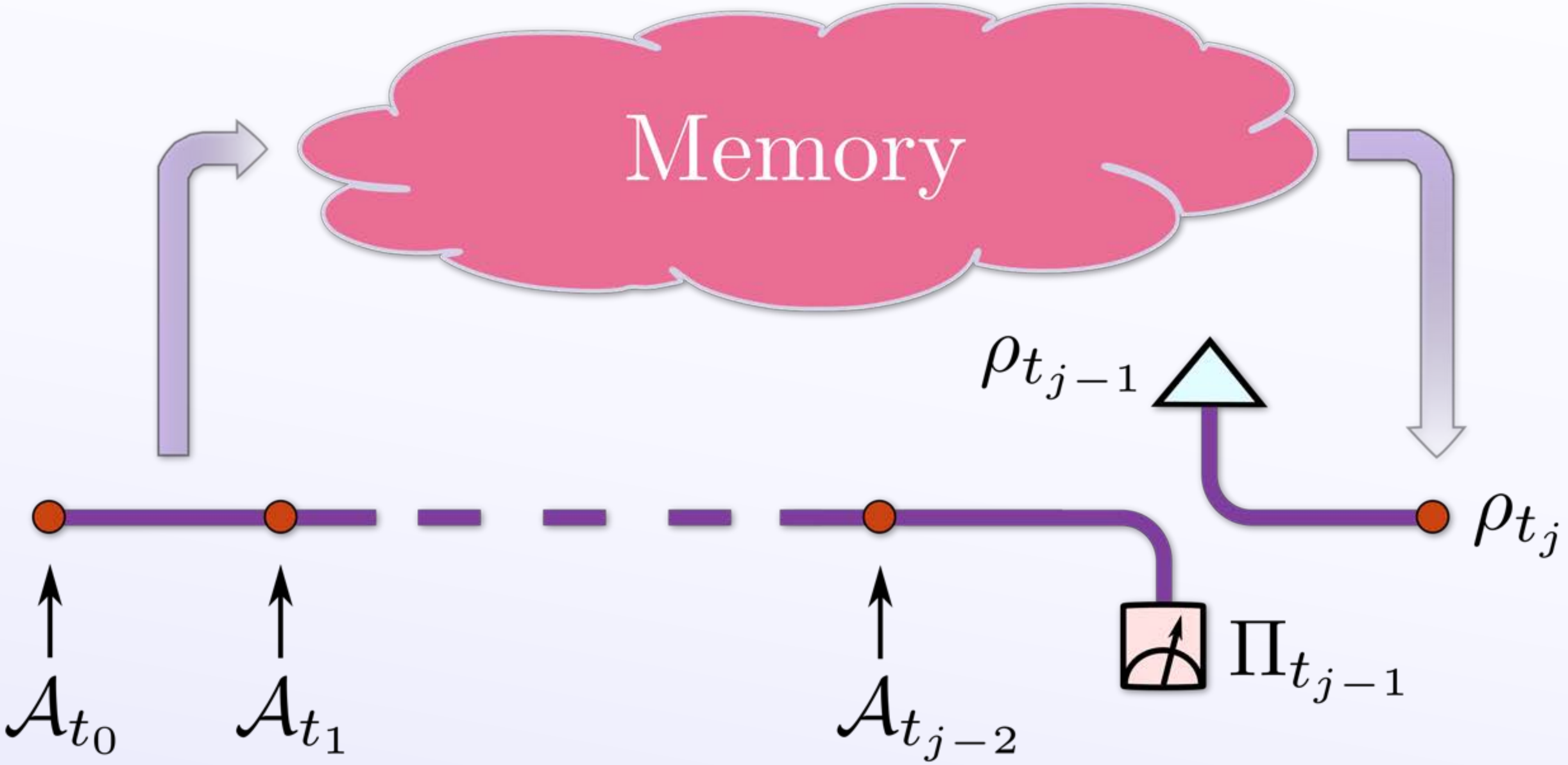

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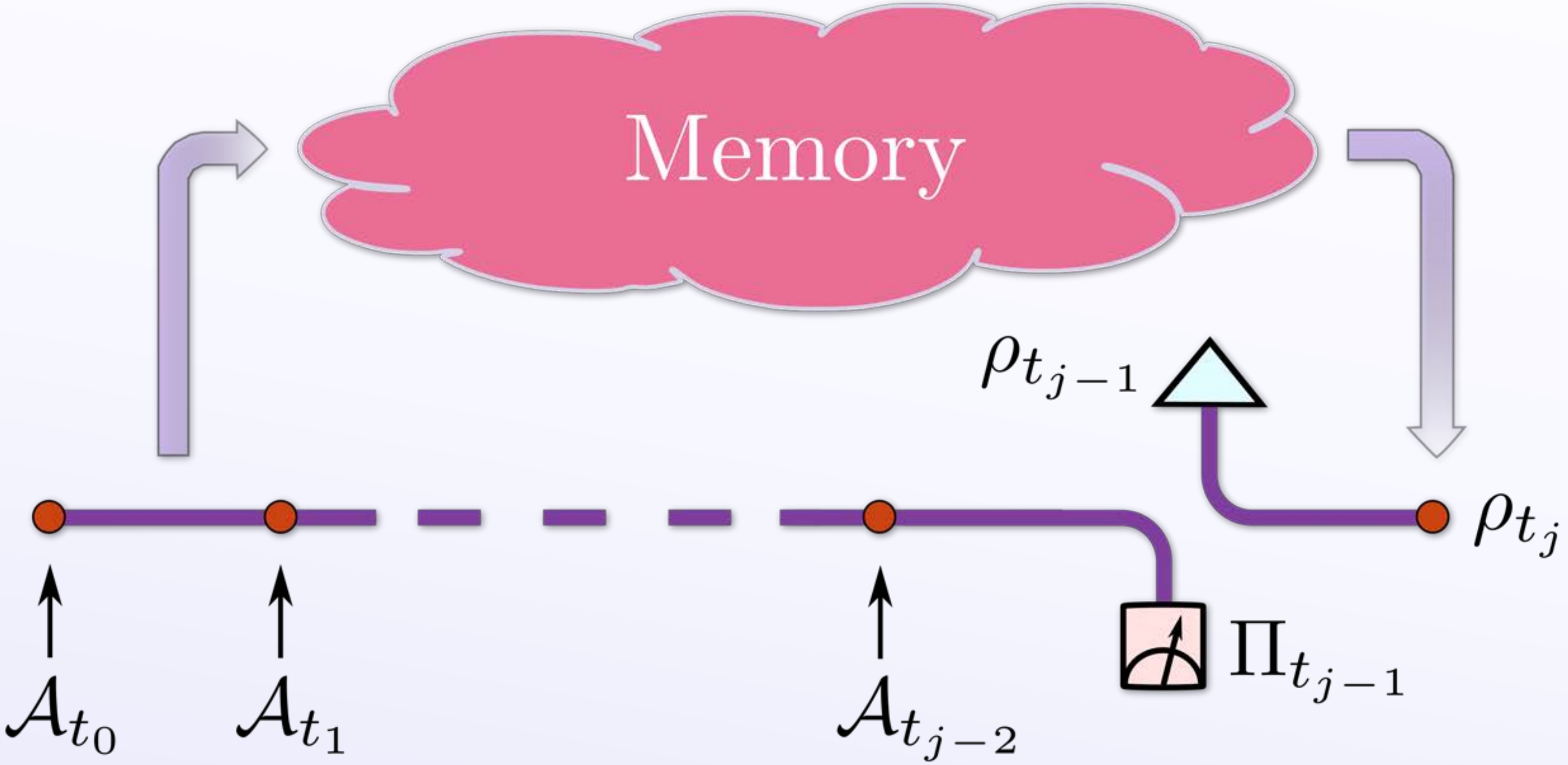
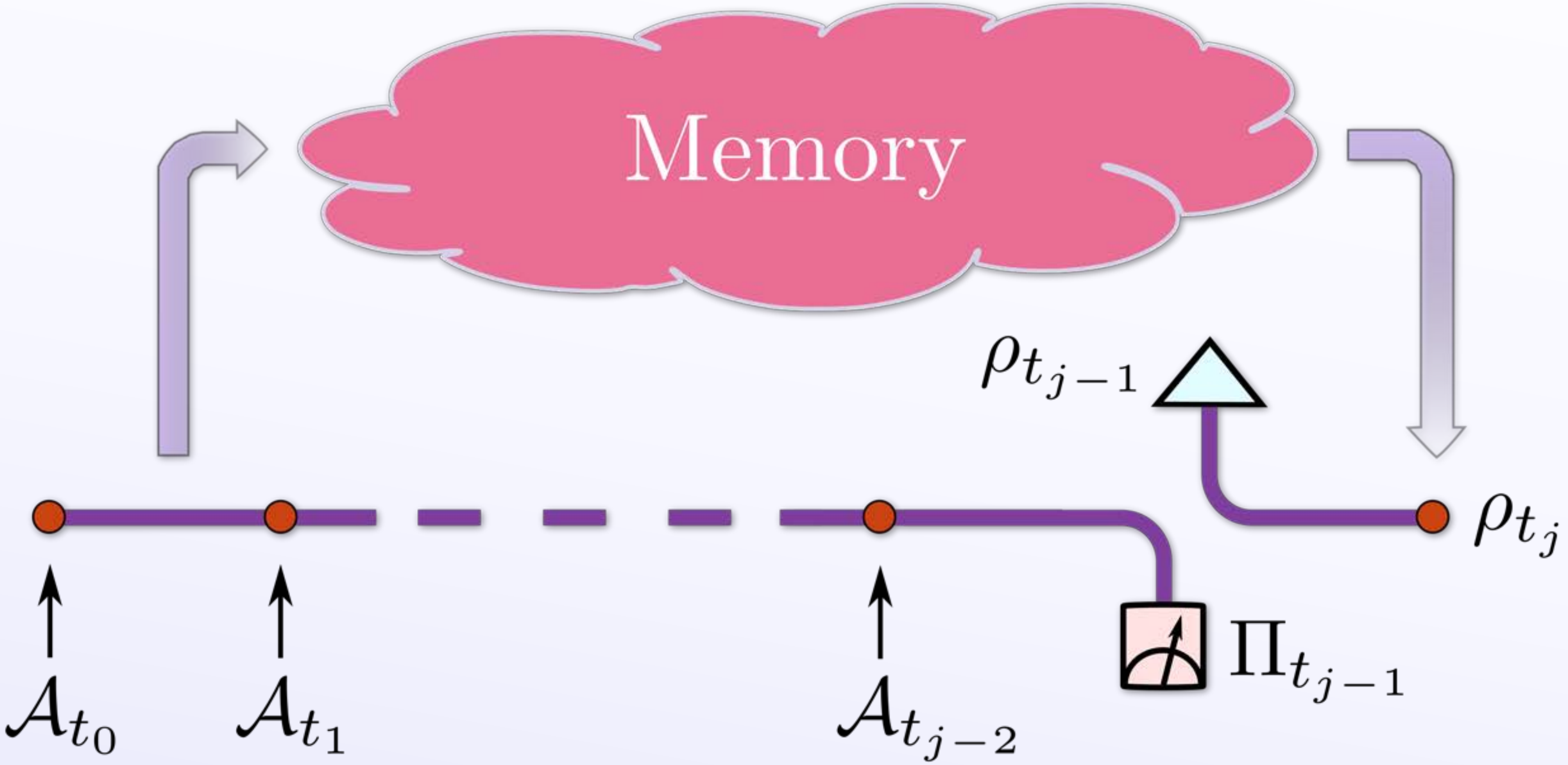


Markovian iff for all choices of control operation:

$$\mathcal{T}_{t_j:t_0}[\mathcal{A}_{t_0}, \mathcal{A}_{t_1}, \cdots, (\Pi_{t_{j-1}}, \rho_{t_{j-1}})] = \mathcal{T}_{t_j:t_0}[\mathcal{A}'_{t_0}, \mathcal{A}'_{t_1}, \cdots, (\Pi'_{t_{j-1}}, \rho_{t_{j-1}})]$$

Process tensor description

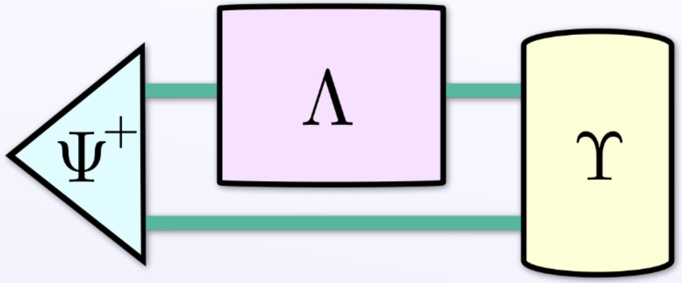


Markovian iff for all choices of control operation:

$$\rho_{t_j}(\rho_{t_{j-1}} | \mathcal{A}_{t_0}, \mathcal{A}_{t_1}, \cdots, \mathcal{A}_{t_{j-2}}, \Pi_{t_{j-1}}) = \rho_{t_j}(\rho_{t_{j-1}})$$

Process tensor description

State representation – Choi-Jamiołkowski isomorphism

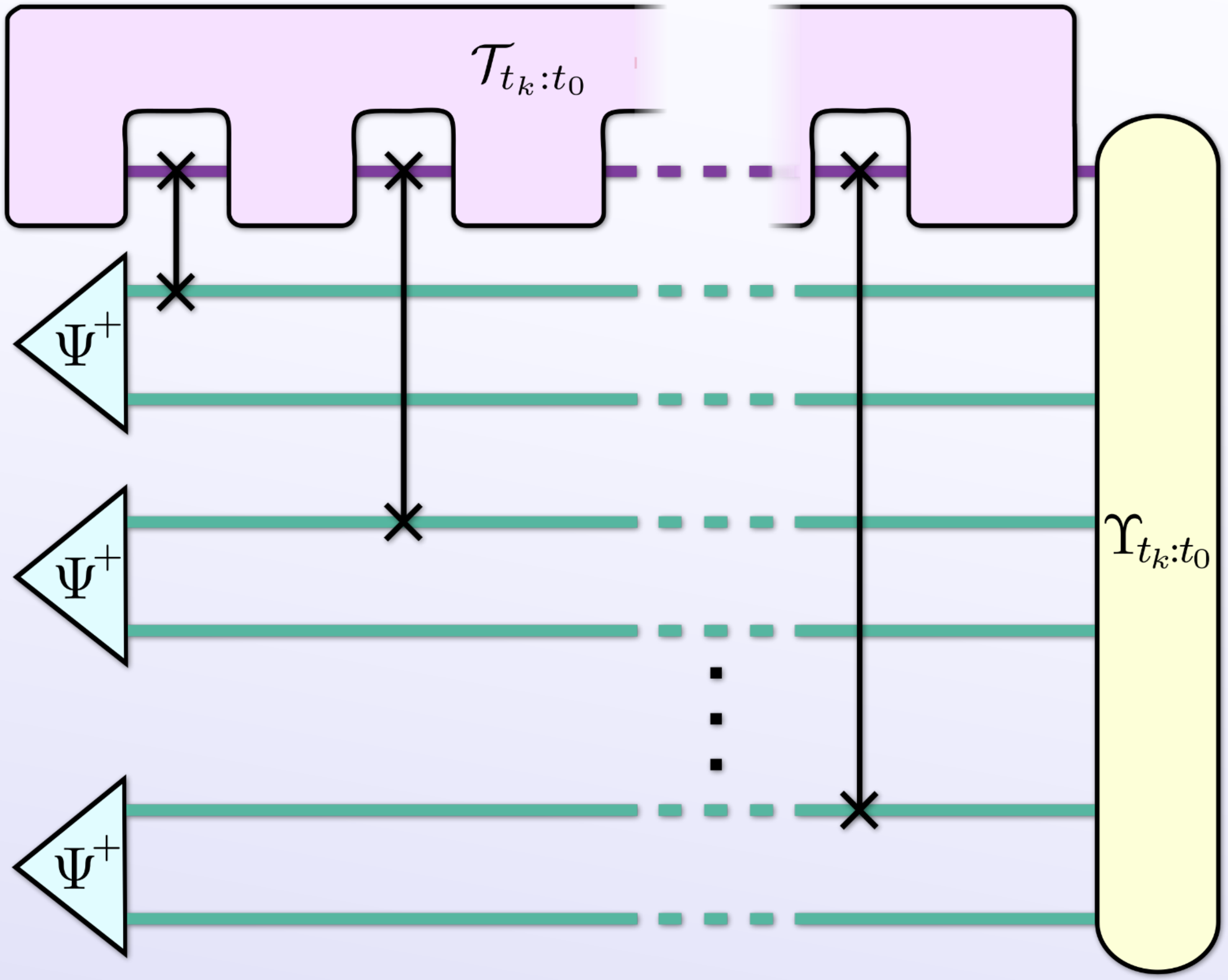


$$|\Psi^+\rangle = \sum_j \frac{1}{\sqrt{d}}|jj\rangle$$

$$\Lambda \leftrightarrow \Upsilon$$

Process tensor description

State representation – correlations in time mapped to correlations in space



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$$\mathcal{T}_{k:0} \leftrightarrow \Upsilon_{k:0}$$

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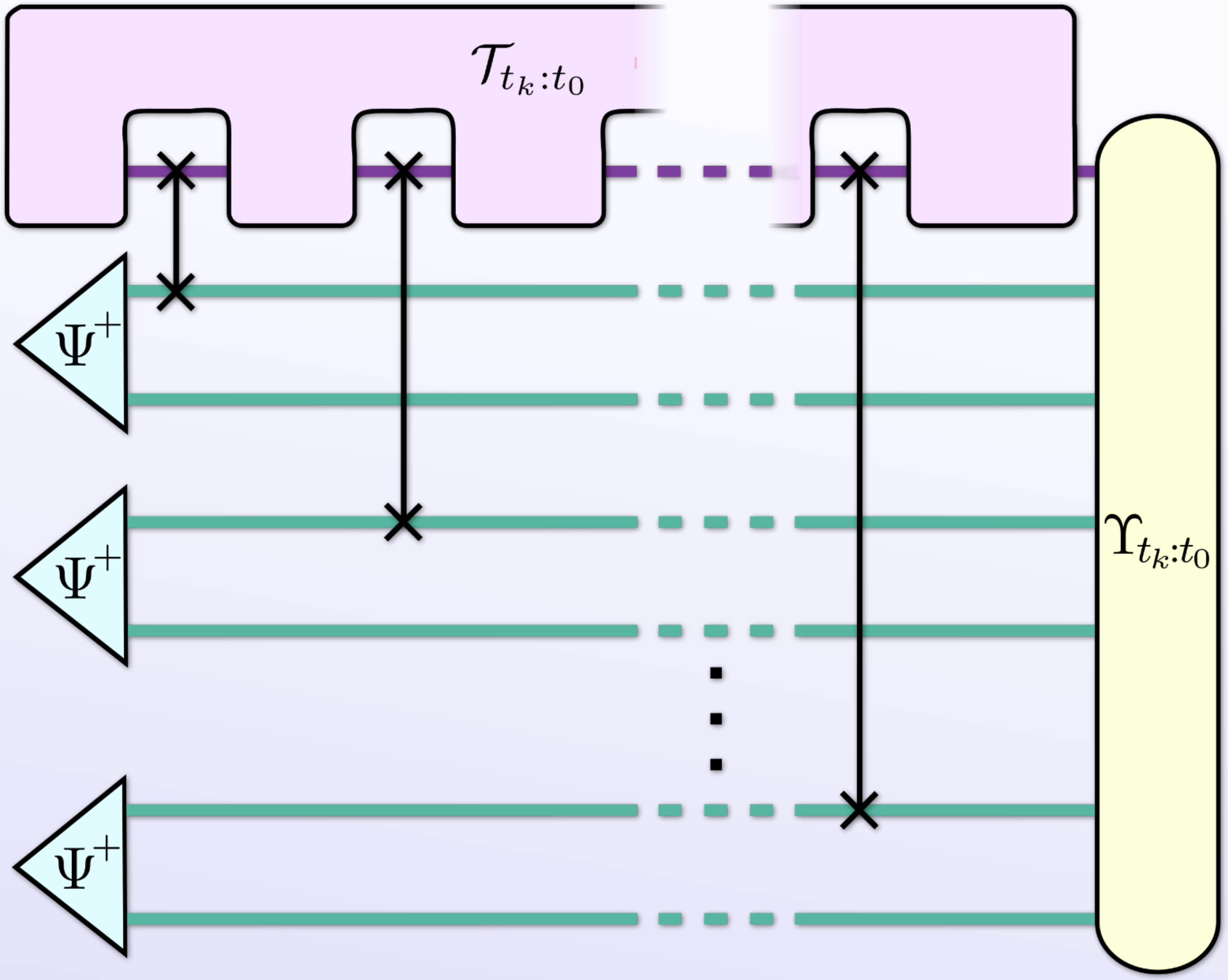

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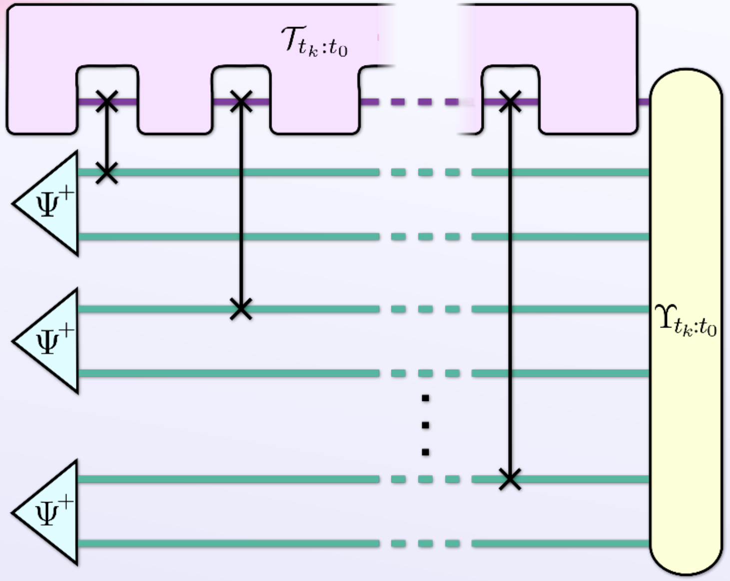
$2k+1$ subsystems

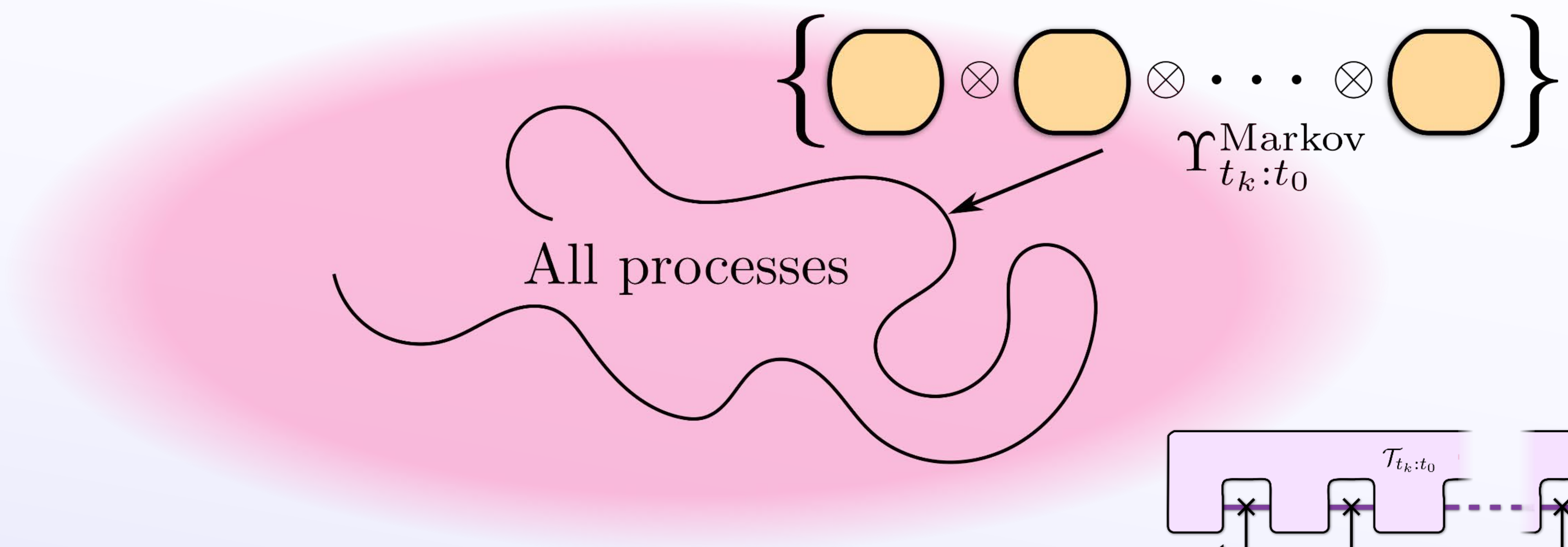
~Natural MPO form

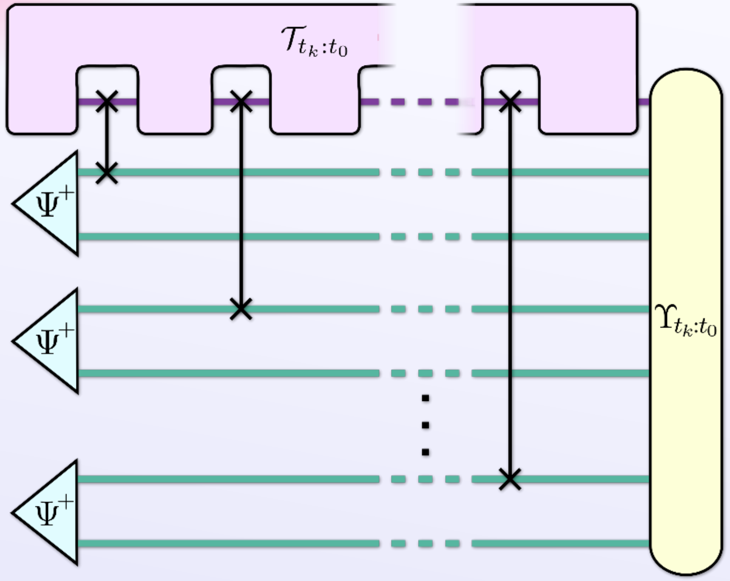
A measure for non-Markovianity

A MEASURE FOR NON-MARKOVIANITY



A measure for non-Markovianity





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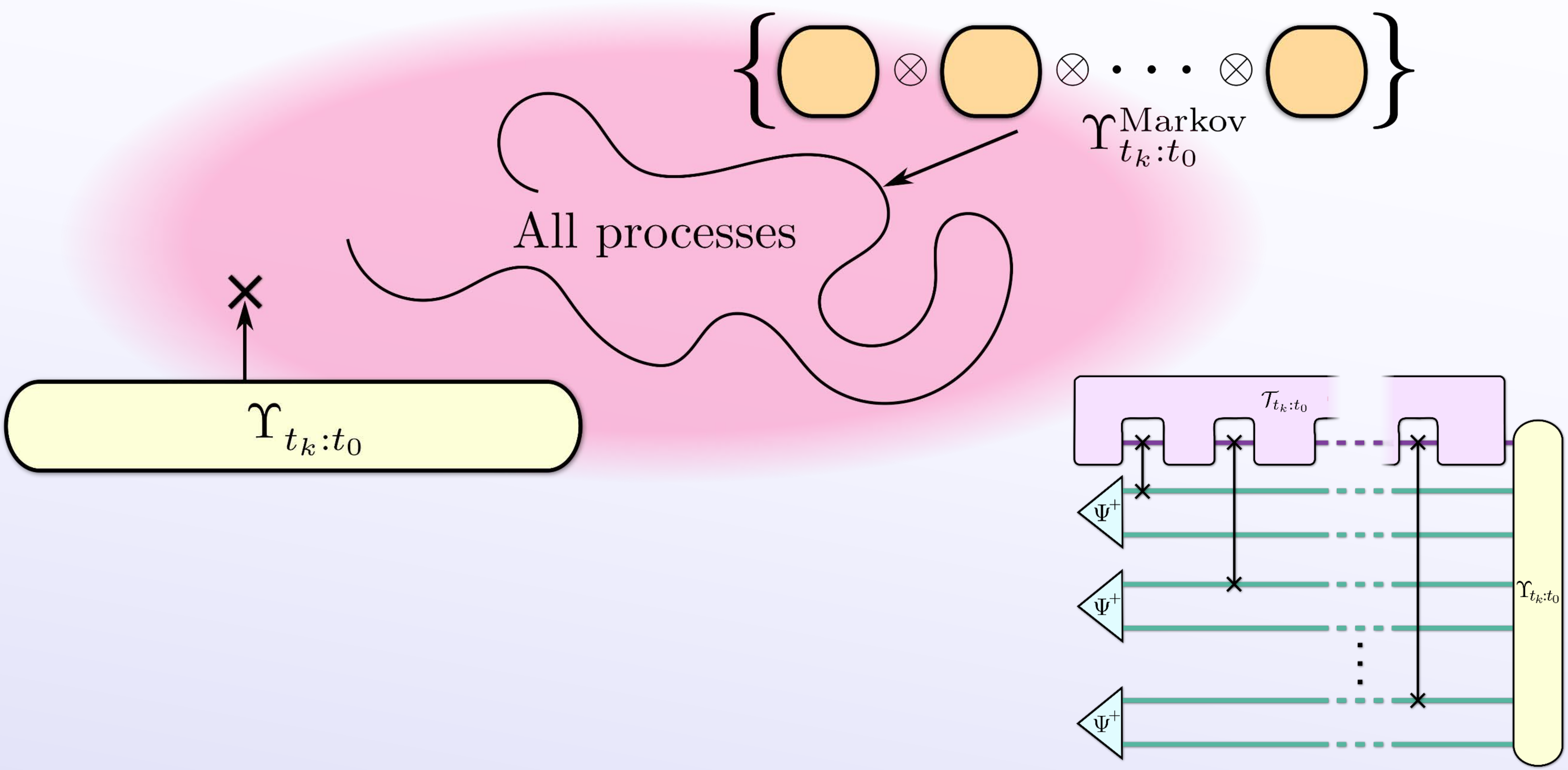

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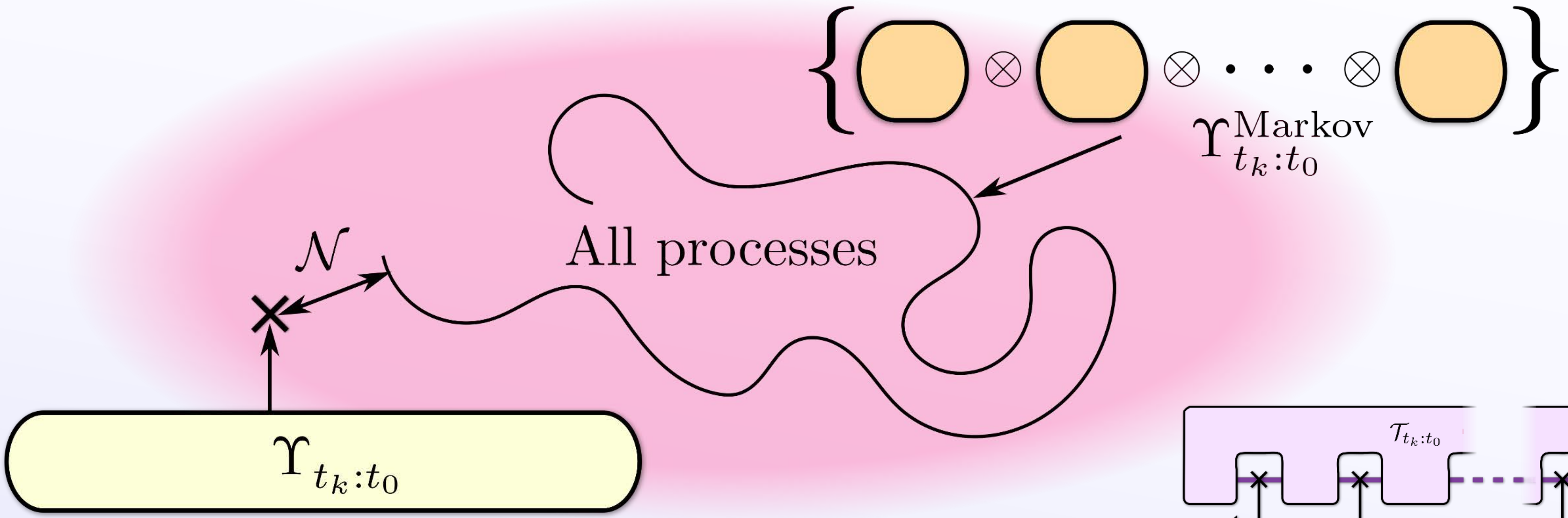
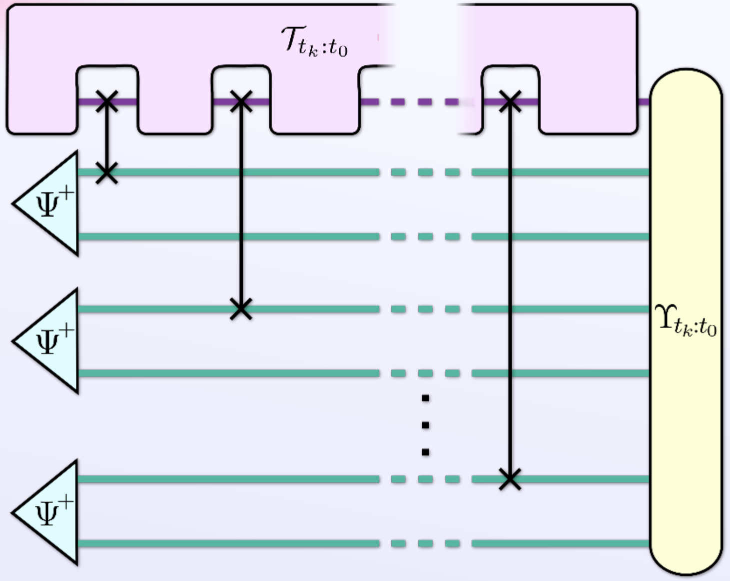
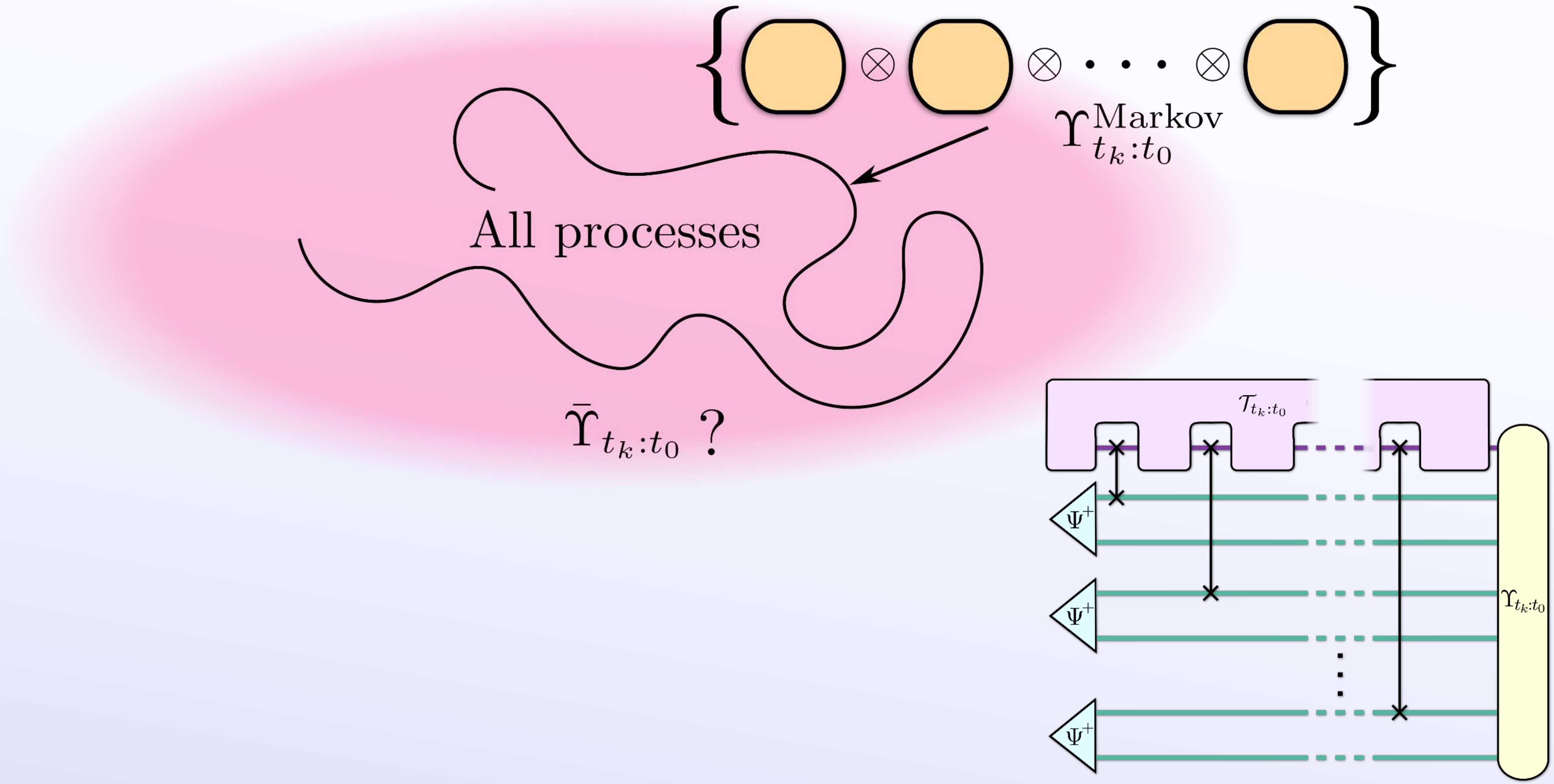


$$\mathcal{N} := \min_{\Upsilon^{\mathrm{Markov}}_{k:0}} \mathcal{D}\left[\Upsilon_{k:0} \| \Upsilon^{\mathrm{Markov}}_{k:0}\right]$$



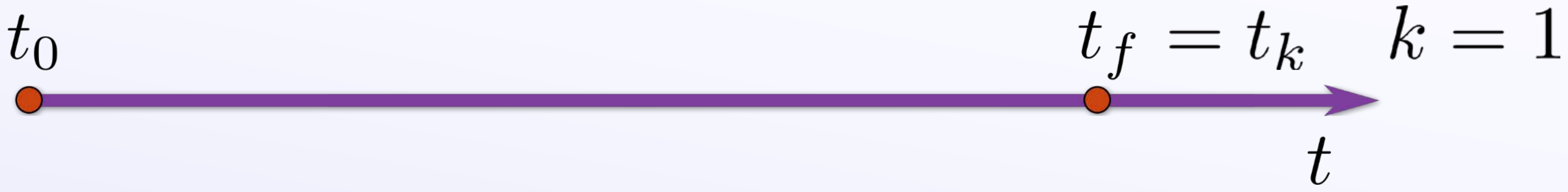
Statistics of quantum processes?



Continuous time description

Is there a general equation of motion for $\mathcal{T}_{t_k:t_0}$?

$$\partial_{t_k} \mathcal{T}_{t_k:t_0}[\mathbf{A}_{t_{k-1}:t_0}] = ?$$



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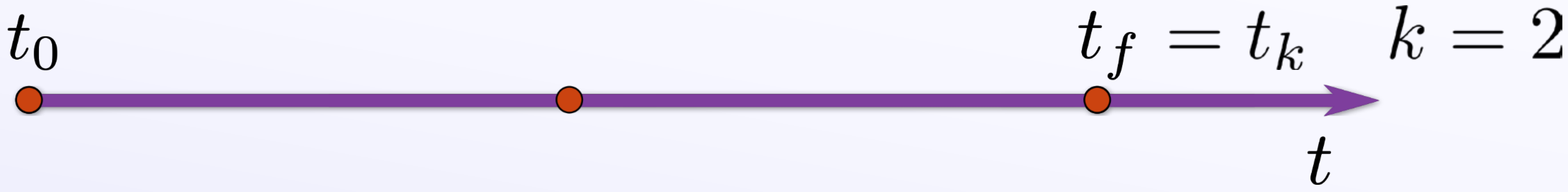

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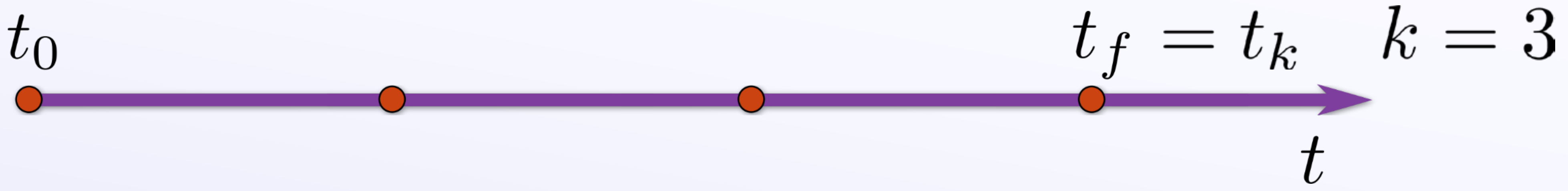

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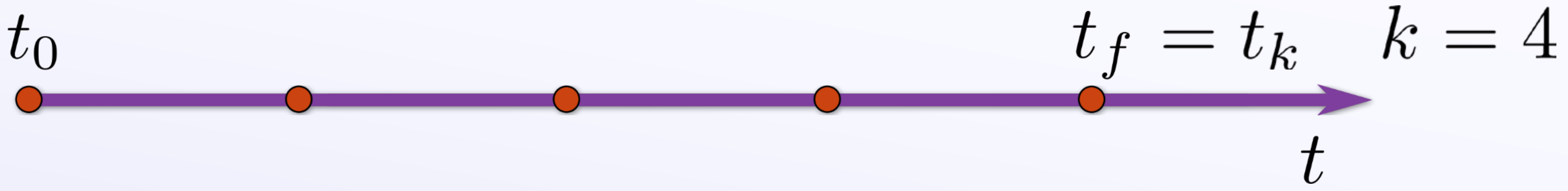

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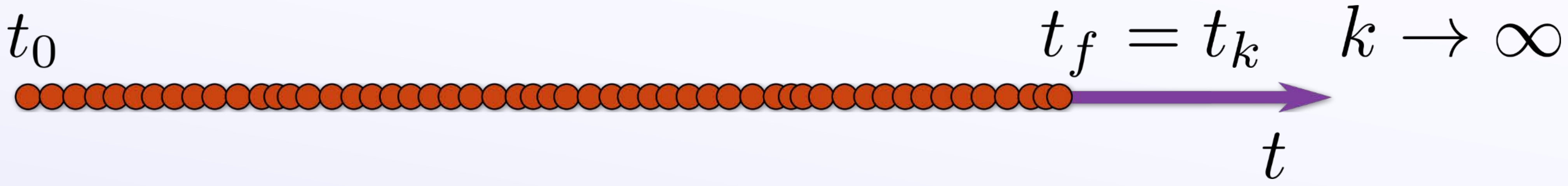
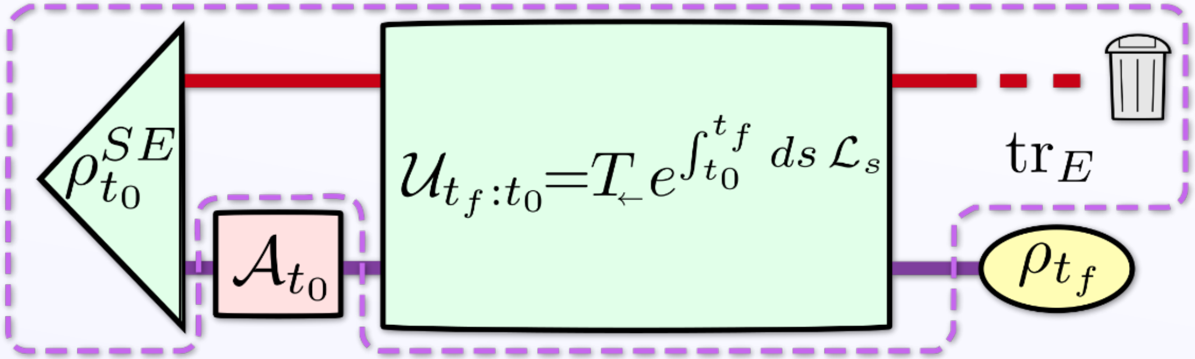


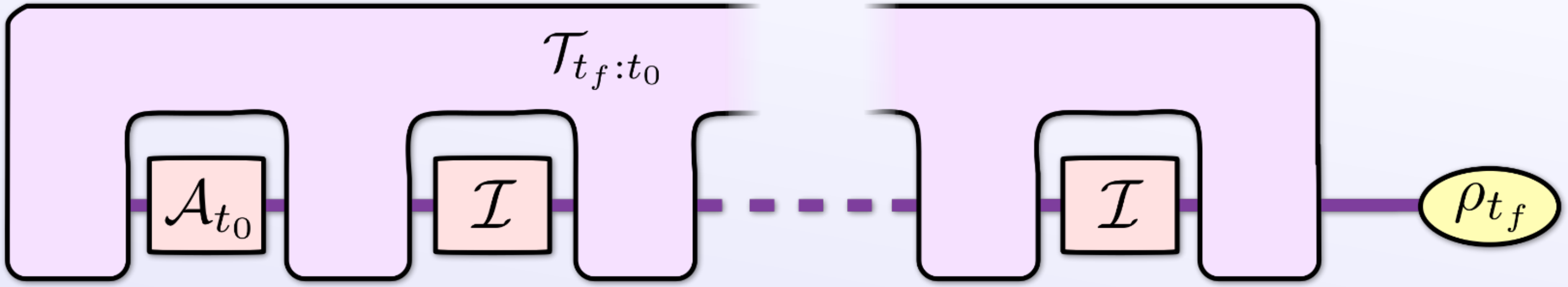
Generalisation of Kolmogorov's extension theorem (Accardi et al 1982) means that this limit is sensible.

Continuous time description

In the 'usual' case, with only initial preparation:



$\Updownarrow$

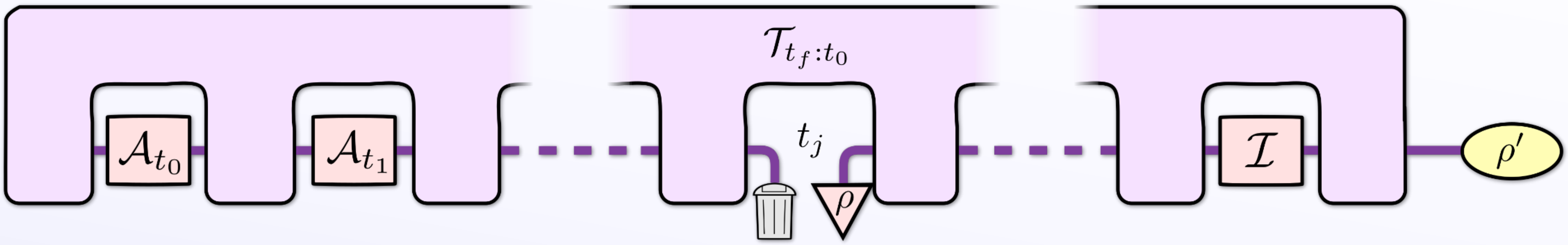


$$\mathcal{I}\rho = \rho$$

$$\partial_t \rho_t^{SE} = \mathcal{L}_t \rho_t^{SE} \overset{\text{e.g.}}{=} -\frac{i}{\hbar}[H_t^{SE}, \rho_t^{SE}]$$

Continuous time description

Intermediate dynamics:



Continuous time description

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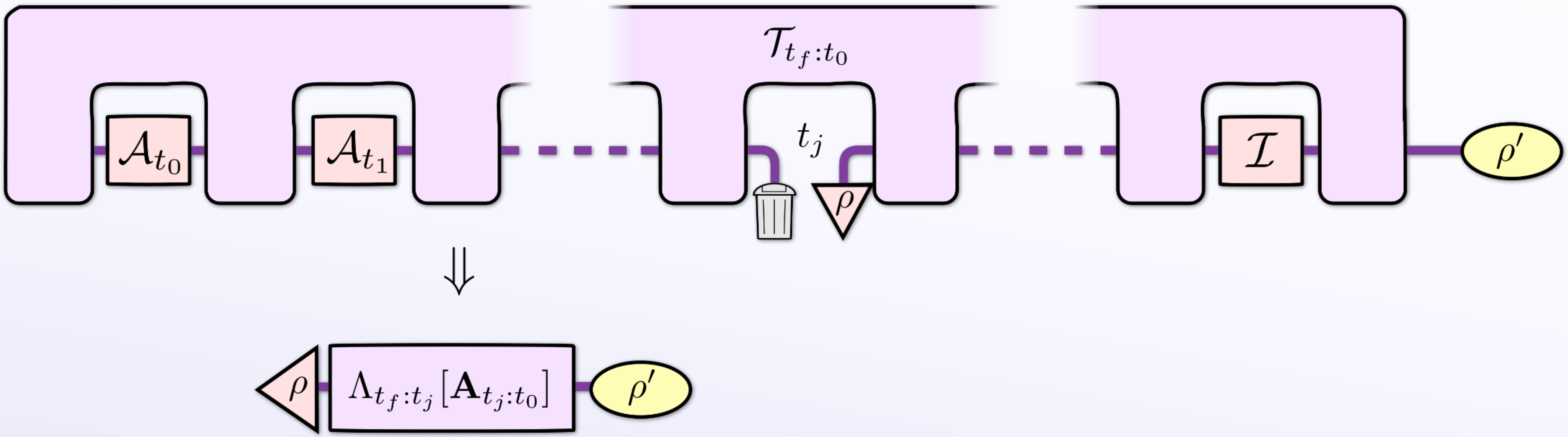

Continuous time description

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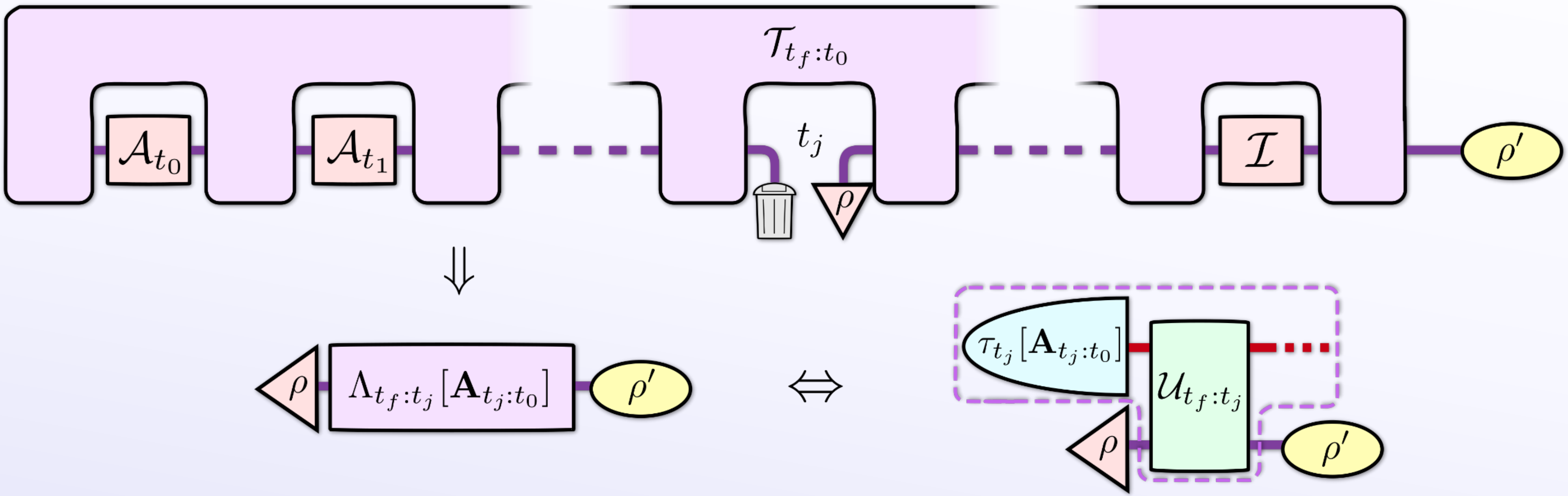
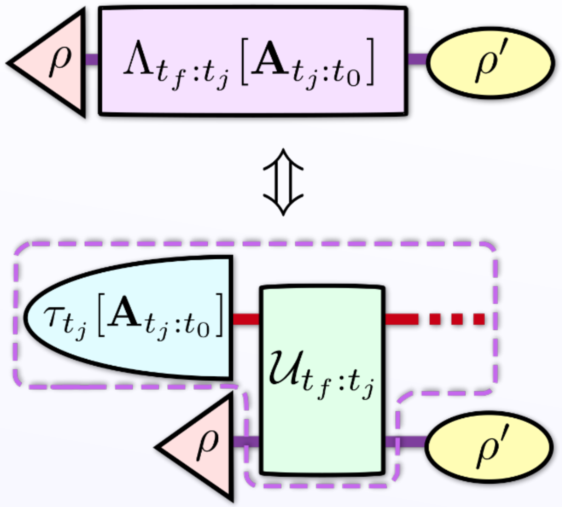


Continuous time description

Break up the evolution:

$$\rho_{t_f} = \Lambda_{t_f:s}\rho_s + (\rho_{t_f} - \Lambda_{t_f:s}\rho_s) = (\mathbf{P}_s + \mathbf{Q}_s)\rho_{t_f}$$

$$\rho_{t_f} =(\mathbf{P}_{t_{k-1}} + \mathbf{Q}_{t_{k-1}})\rho_{t_f}$$



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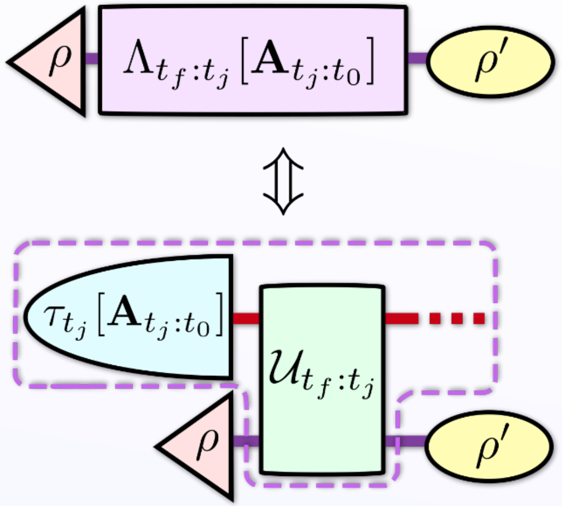

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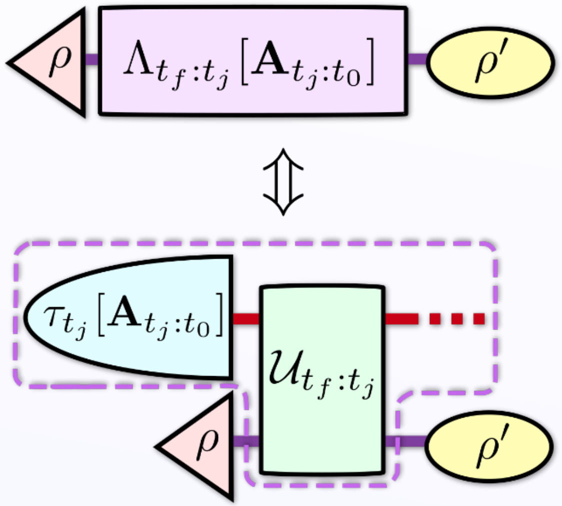

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$$\rho_{t_f} = (\mathbf{P}_{t_{k-1}} + \mathbf{Q}_{t_{k-1}}(\mathbf{P}_{t_{k-2}} + \mathbf{Q}_{t_{k-2}} (\mathbf{P}_{t_{k-3}} + \mathbf{Q}_{t_{k-3}})))\rho_{t_f}$$

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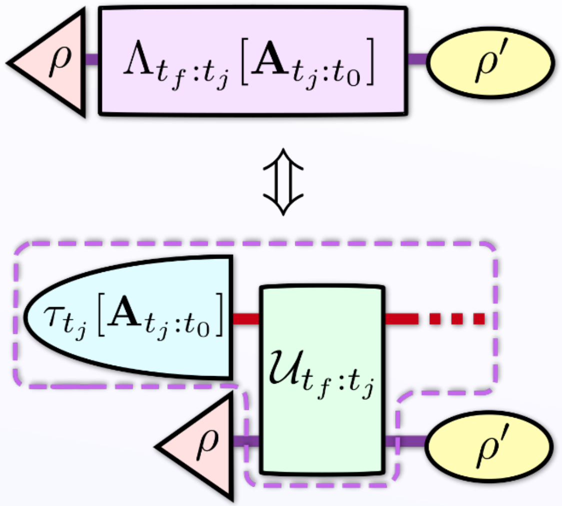

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$$\begin{aligned}\rho_{t_f} =&(\mathbf{P}_{t_{k-1}} + \mathbf{Q}_{t_{k-1}}\mathbf{P}_{t_{k-2}} + \mathbf{Q}_{t_{k-1}}\mathbf{Q}_{t_{k-2}}\mathbf{P}_{t_{k-3}} \\ &+ \cdots + \mathbf{Q}_{t_{k-1}}\mathbf{Q}_{t_{k-2}}\ldots\mathbf{Q}_{t_1}\mathbf{P}_{t_0} \\ &+ \mathbf{Q}_{t_{k-1}}\mathbf{Q}_{t_{k-2}}\ldots\mathbf{Q}_{t_0})\rho_{t_f}\end{aligned}$$

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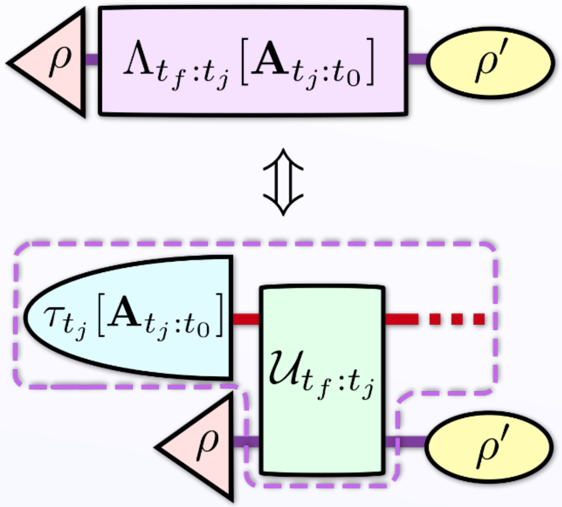

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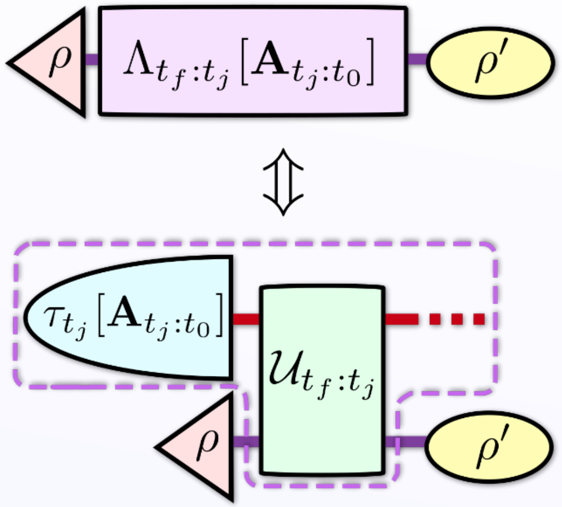
$$\begin{aligned}\mathbf{Q}_{t_{k-1}}\mathbf{Q}_{t_{k-2}}\cdots\mathbf{Q}_{t_{j+1}}\mathbf{P}_{t_j}\rho_{t_k} &= T^{(k-j)}_{t_k:t_j}\rho_{t_j} = \Lambda_{t_k:t_j} - \sum_{l=j+1}^{k} T^{(k-l)}_{t_k:t_l}\Lambda_{t_l:t_j} \\ &= \Lambda_{t_k:t_j} - \sum_l \Lambda_{t_k:t_l}\Lambda_{t_l:t_j} + \sum_l \sum_{m<l} \Lambda_{t_k:t_l}\Lambda_{t_l:t_m}\Lambda_{t_m:t_j} - \ldots\end{aligned}$$

Continuous time description

Break up the evolution:



$$\rho_{t_f} = \Lambda_{t_f:s}\rho_s + (\rho_{t_f} - \Lambda_{t_f:s}\rho_s) = (\mathbf{P}_s + \mathbf{Q}_s)\rho_{t_f}$$

$$\begin{aligned}\rho_{t_f} =& (\mathbf{P}_{t_{k-1}} + \mathbf{Q}_{t_{k-1}}\mathbf{P}_{t_{k-2}} + \mathbf{Q}_{t_{k-1}}\mathbf{Q}_{t_{k-2}}\mathbf{P}_{t_{k-3}} \\ &+ \cdots + \mathbf{Q}_{t_{k-1}}\mathbf{Q}_{t_{k-2}}\ldots\mathbf{Q}_{t_1}\mathbf{P}_{t_0} \\ &+ \mathbf{Q}_{t_{k-1}}\mathbf{Q}_{t_{k-2}}\ldots\mathbf{Q}_{t_0})\rho_{t_f}\end{aligned}$$

Transfer tensor
(cf. Cerrillo & Cao, 2014)

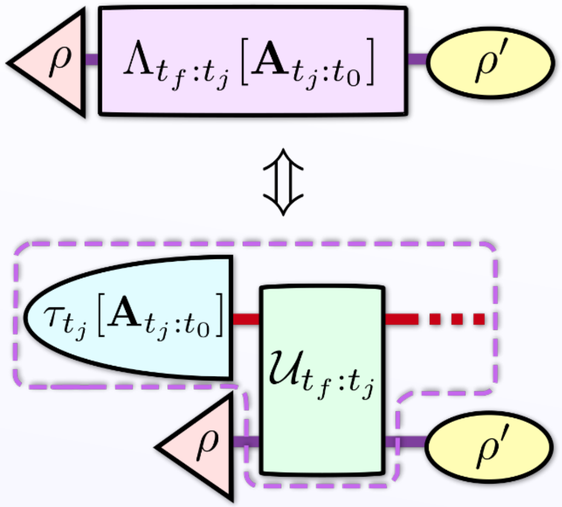
$$\mathbf{Q}_{t_{k-1}}\mathbf{Q}_{t_{k-2}}\cdots\mathbf{Q}_{t_{j+1}}\mathbf{P}_{t_j}\rho_{t_k} = T^{(k-j)}_{t_k:t_j}\rho_{t_j} = \Lambda_{t_k:t_j} - \sum_{l=j+1}^{k} T^{(k-l)}_{t_k:t_l}\Lambda_{t_l:t_j}$$

$$= \Lambda_{t_k:t_j} - \sum_{l}\Lambda_{t_k:t_l}\Lambda_{t_l:t_j} + \sum_{l}\sum_{m<l}\Lambda_{t_k:t_l}\Lambda_{t_l:t_m}\Lambda_{t_m:t_j} - \ldots$$

Continuous time description

Break up the evolution:

$$\frac{\rho_{t_k} - \rho_{t_{k-1}}}{\delta t} = \frac{(\Lambda_{t_k:t_k-\delta t} - \mathcal{I})\rho_{t-\delta t}}{\delta t} + \frac{1}{\delta t^2}\sum_{j=0}^{k-2} \delta t\, T^{(k-j)}_{t_k:t_j}\rho_{t_j}$$
$$+ \frac{1}{\delta t}\mathbf{Q}_{t_{k-1}}\mathbf{Q}_{t_{k-2}}\dots\mathbf{Q}_{t_0}\rho_{t_k}.$$



Continuous time description

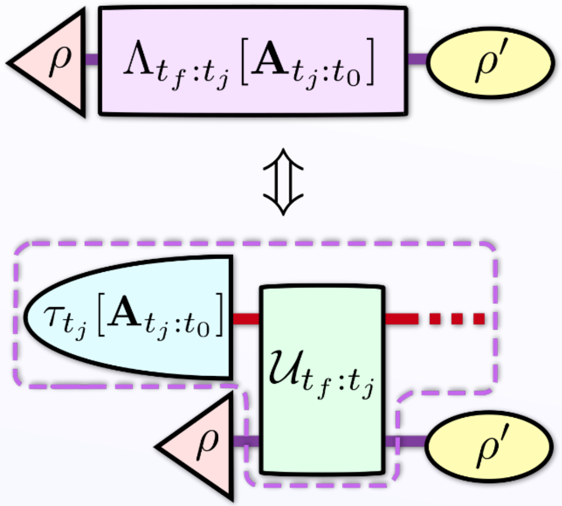
Break up the evolution:

$$\frac{\rho_{t_k} - \rho_{t_{k-1}}}{\delta t} = \frac{(\Lambda_{t_k:t_k-\delta t} - \mathcal{I})\rho_{t-\delta t}}{\delta t} + \frac{1}{\delta t^2}\sum_{j=0}^{k-2} \delta t\, T^{(k-j)}_{t_k:t_j}\rho_{t_j}$$

$$+ \frac{1}{\delta t}\mathbf{Q}_{t_{k-1}}\mathbf{Q}_{t_{k-2}}\dots\mathbf{Q}_{t_0}\rho_{t_k}.$$

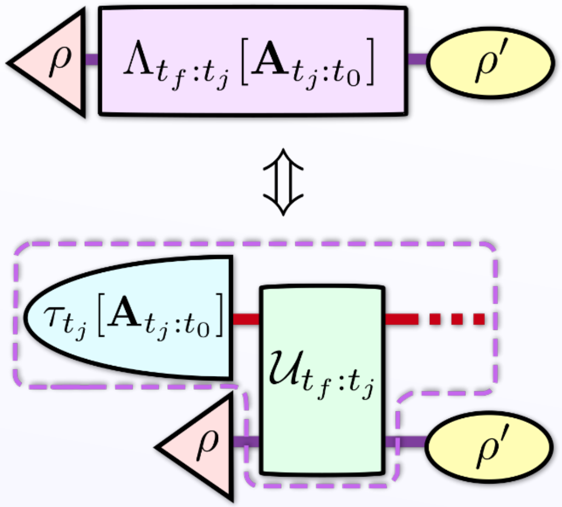
$$\Big\downarrow\; k \to \infty$$

$$\partial_t \rho_t = \mathcal{L}^S_t \rho_t + \int_{t_0}^{t} ds\, \mathcal{K}_{t,s}\rho_s + \mathcal{J}_{t,t_0}$$



Continuous time description

Break up the evolution:



$$\frac{\rho_{t_k} - \rho_{t_{k-1}}}{\delta t} = \frac{(\Lambda_{t_k:t_k-\delta t} - \mathcal{I})\rho_{t-\delta t}}{\delta t} + \frac{1}{\delta t^2}\sum_{j=0}^{k-2} \delta t\, T^{(k-j)}_{t_k:t_j}\rho_{t_j}$$

$$+ \frac{1}{\delta t}\mathbf{Q}_{t_{k-1}}\mathbf{Q}_{t_{k-2}}\dots\mathbf{Q}_{t_0}\rho_{t_k}.$$

Time-local evolution

$$\Big\downarrow\; k \to \infty$$

$$\partial_t \rho_t = \mathcal{L}^S_t \rho_t + \int_{t_0}^{t} ds\, \mathcal{K}_{t,s}\rho_s + \mathcal{J}_{t,t_0}$$

Nakajima Zwanzig equation

Memory kernel

Inhomogeneous term

Summary

- Complete framework for characterising non-Markovian processes.
- Efficient state representation: temporal correlations become spatial.
- Operationally meaningful definition of non-Markovianity.
- Sensible continuous time limit: 'tomographic master equations'.